1 *Logical Connectives*

Now that we have ways of building simple expressions using predicates, terms, and quantifiers, we should naturally ask how we can express more complicated ideas such as the following.

- *"The number* 1 *is not prime, but every prime number greater than* 2 *is odd."*
- *"Every person of taste appreciates the Mazda Miata."*
- *"If Alcaraz doesn't take the first set against Sinner, then Sinner will finish undefeated."*

We can think of each of these sentences as *"compound expressions"* that are formed by taking simpler expressions—represented by some number of quantifiers and predicates applied to terms—and connecting them together to produce a final truth value. This requires us to define a collection of *connectives*[1] for manipulating expressions according to their truth values.

[1] These are somtimes called *logical connectives* because they connect together complete logical expressions. They are sometimes also called *sentential connectives* or *propositional connectives* since they connect *sentences* together.

In the same way that we might think of predicates as *"functions"* that transform *objects* into *truth values*, we can think of logical connectives as *"functions"* that transform *truth values* into other *truth values.* Since we only have two truth values, we can define an n-ary connective by specifying, for each possible arrangement of n input truth values, which of $\top$ or $\bot$ to produce as the output truth value.[2] We can easily represent these associations in a table, which we appropriately refer to as a ***truth table***. Due to the abusive treatment of truth tables in other contexts, we emphasize that *truth tables* ***only*** *serve to help us* ***define*** *the logical connectives*. They do not play any role in the art of proof writing, and in fact they are not strictly necessary to include at any point in the development of our logical system (they only serve as a concise and intuitive visual for understanding how the logical connectives might be interpreted as functions that map between truth values).

[2] Throughout this article, we will sometimes use the notation $x \overset{f}{\mapsto} y$ to denote a function named f which maps a given input x to the output y. The more common syntax $f(x) = y$ that you have surely encountered until this point conveys basically the same thing; however, this syntax has the advantage of allowing us to write $x \mapsto y$ to denote an *anonymous* function mapping the input x to the output y. This is only convenient (and never necessary) in those situations when we want to talk about a mapping without having to give it a name, such as at the beginning of the next section.

For the remainder of this discussion, we fix a universe of discourse consisting of all of the natural numbers 0, 1, 2, … and introduce the following predicates.

$$\begin{aligned} >(x,y) &\coloneqq \text{“}x \textit{ is greater than } y.\text{”} \\ \geqslant(x,y) &\coloneqq \text{“}x \textit{ is greater than or equal to } y.\text{”} \\ \nu(x) &\coloneqq \text{“}x \textit{ is a natural number.”} \\ \pi(x) &\coloneqq \text{“}x \textit{ is prime.”} \\ \varepsilon(x) &\coloneqq \text{“}x \textit{ is even.”} \\ o(x) &\coloneqq \text{“}x \textit{ is odd.”} \end{aligned}$$

1.1 *Negations*

Table 1. A truth table defining the *negation* $\neg p$ of a formula p.

p	$\neg p$
$\top$	$\bot$
$\bot$	$\top$

The simplest kind of logical connective we could imagine would be a unary function $x \mapsto x$ whose output is the same its unmodified input. There is obviously no need for introducing syntax for such a connective because it doesn't really do anything. The next-simplest logical connective then might be a unary connective that *inverts* its input, mapping each truth value input to the *other* one as its output. If p represents
$\neg$ a well-formed sentence with a truth value, then we denote by $\neg p$ the sentence with the opposite truth value. This is described in Table 1. Equipped with this first logical connective, we could now express a thought like *"*3 *is not even"* by saying "$\neg\varepsilon(3)$."[3]

[3] You might think this means the same thing as *"*3 *is odd,"* and that we therefore could have just said "$o(3)$," but those sentences don't (necessarily) mean the same thing! We will soon discover that the complementary relationship between even and odd that might seen very natural and obvious actually requires quite a bit of proof.

1.2 Junctions

The other unary connectives are not particularly interesting, and it seems weird to call them *"connectives"* anyways since they don't really *"connect"* anything together; you need at least *two* things to connect! The rest of our logical connectives will be binary, and the simplest way to connect two sentences together should be to just *assert both at the same time.* This is usually done by the English word *"and,"* and we $\wedge$ will call the associated connective the ***conjunction*** and use the symbol $\wedge$ to denote it. We define this connective in Table 2. For example, if you wanted to communicate the sentence *"3 is odd and 3 is prime,"* you could write *"$o(3) \wedge \pi(3)$."*

Table 2. A truth table defining the *conjunction* $p \wedge q$ and *disjunction* $p \vee q$ of two formulae p and q.

p	q	$p \wedge q$	$p \vee q$
$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$

Alternatively, given two sentences we want to connect together, we may decide that the *overall* phrase should be true without necessarily asserting that *both* of the sentences are themselves *individually true.* Using the English word *"or,"* we can express that *at least one* of the sentences is *true* even if we weren't sure which one it $\vee$ was. We call this kind of connective a ***disjunction*** and denote it with the $\vee$ symbol. As before, this connective is defined in Table 2.

1.3 Conditionals

Despite not seeming as *"basic"* or *"simple"* as the $\wedge$ and $\vee$, the kind of reasoning that our next connective encodes is by far the most common kind of unquantified expression in everyday mathematical speech. To motivate this discussion, consider the following sentence.

› *"Every prime number greater than 2 is odd."*

If our universe of discourse only contained prime numbers greater than 2, then $\forall x(o(x))$ would be a straightforward symbolization of that sentence; but, our universe of discourse contains *all* of the natural numbers. In order to formalize this sentence, we need a way to assert the predicate $o(\cdot)$ *only* for those natural numbers that satisfy both $\pi(\cdot)$ and $>(\cdot, 2)$ *without asserting anything* for those natural numbers that fail to satisfy both conditions. One natural way to communicate this in English would be with a *conditional* expression.

› *"For every natural number x, if x is prime and x is greater than 2, then x is odd."*

The core part of that expression, *"if x is prime and x is greater than 2, then x is odd,"* should be interpreted to assert *"x is odd"* when the condition *"x is prime and x is greater than 2"* is *true* but *not to assert anything* when that condition is *false.* This idea is captured by our next connective—***material implication***—which we use the symbol $\rightarrow$ $\rightarrow$ to denote. In this example, our *"premise"* is $\pi(x) \wedge x > 2$ and our *"conclusion"* is $o(x)$, so we can symbolize our sentence using this new connective as follows.[4]

$$\forall x\Big(\big(\pi(x) \wedge x > 2\big) \rightarrow o(x)\Big)$$

[4] Notice the use of parentheses in the expression. If we had instead written $\forall x(\pi(x) \wedge x > 3 \rightarrow o(x))$, it would be ambiguous whether we meant to say $(\pi(x) \wedge x > 3) \rightarrow o(x)$ for all x or $\pi(x) \wedge (x > 3 \rightarrow o(x))$ for all x. Parentheses are required here because $\wedge$ and $\rightarrow$ do not associate.

An important thing to note: since we only want to assert $o(x)$ when $\pi(x) \wedge x > 2$ is satisfied, we need to be careful in defining what truth values $\rightarrow$ will produce for different inputs. We certainly want $\top \rightarrow \top$ to be $\top$, and we also need $\top \rightarrow \bot$ to be $\bot$. But what do we do about the situations (*i.e.,* $\bot \rightarrow \top$ and $\bot \rightarrow \bot$) where the premise to the conditional is *false?* If we take the idea that these should resolve as $\bot$, then our statement $(\pi(x) \wedge x > 2) \to \pi(x)$ above would be *false* when $x := 1$ because its premise $\pi(1) \wedge 1 > 2$ would be *false*, and that would then cause our entire universal quantification to be *false*. Think about that: do you think that the number 1 is a *counterexample* to the statement *"Every prime number greater than 2 is odd"*? Hopefully, you don't. That's not generally what we understand *"if / then"* to mean. Our only option then is to define $\bot \to \top$ and $\bot \to \bot$ to be $\top$. We can summarize this as an interpretation of $\to$ that is $\bot$ only when its premise is $\top$ and its conclusion is $\bot$; otherwise, we interpret $\to$ to be $\top$. We define this connective in Table 3.

Table 3. A truth table defining the *material conditional* $p \to q$ and *biconditional* $p \leftrightarrow q$ of two formulae p and q.

p	q	$p \to q$	$p \leftrightarrow q$
$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\top$	$\top$

We should be careful to note that conditional expressions can manifest in many different and sometimes unintuitive ways in natural English. Below, we list several equivalent ways to express $p \to q$ in English, where $p :=$ *"Espresso is nourishing"* and $q :=$ *"Alcaraz drink espresso."*

- *"**If** espresso is nourishing, **then** Alcaraz drinks espresso."*
- *"Alcaraz drinks espresso **if** it is nourishing."*
- *"Espresso is **only** nourishing **if** Alcaraz drinks it."*
- *"Espresso is nourishing **only if** Alcaraz drink it."*
- *"Alcaraz drinking espresso **requires** that espresso is nourishing."*
- *"For Alcaraz to drink espresso, it is **sufficient** that espresso is nourishing."*
- *"For espresso to be nourishing, it is **necessary** that Alcaraz drinks it."*
- *"Alcaraz drinks espresso **unless** espresso is not nourishing."*

Because conditional statements are so common in mathematics, in formal logic, and in general, there are many different ways to refer to the components p and q in an expression like $p \to q$, just as there are many synonyms for *"conditional."* We list some examples in Table 4 below.

p	$\to$	q
IMPLICANT	IMPLICATION	IMPLICAND
PREMISE	IMPLICATION	INFERENCE
ANTECEDENT	SEQUENT	CONSEQUENT
PRESCEDENT	SEQUENT	CONSEQUENT
CONDITION	CONDITIONAL	CONCLUSION
ASSUMPTION		CONCLUSION
SUPPOSITION		DEDUCTION
HYPOTHESIS		THESIS

Table 4. Some common and uncommon words used to represent the different components of a statement like $p \to q$. The first and last columns correspond to references for p and q respectively, while the middle column provides some ways to refer to the *entire* conditional $p \to q$ that are consonant with the phrases in that row.