7 Induction and Recursion

Definition 7.1: The Unique Existential Quantifier.
For every *wff* φ with at most one free variable, we define what it means to say *there exists a unique*[1] object that φ is *true* about below.

$\exists!$

$$\exists! x(\varphi(x)) \;:\Leftrightarrow\; \exists x\Big(\varphi(x) \land \forall y\big(\varphi(y) \;\Rightarrow\; y = x\big)\Big)$$

Given a set X, we can extend this definition in a similar way to *restricted quantifier notation*[2] to make complicated sentences more convenient to express.

$$(\exists! x_1, \ldots x_n \in X)(\varphi(x)) \;:\Leftrightarrow\; \exists! x_1 \ldots x_n\big((x_1 \in X \land \ldots \land x_n \in X) \land \varphi(x)\big)$$

[1] Notice that the word *"unique"* means *"only one"*; this is a *unary* predicate that says x is the *only* thing with some given property. It ***does not*** mean the same as the word *"distinct,"* which is a *binary* predicate indicating that $x \neq y$.

[2] *Definition 6.1* from *clavicula 6*

7.1 What is a Function?

Definition 7.2: Function.
function Let X and Y be sets. We define what it means for f to be a *function from X to Y* below.

$f : X \to Y$

$$f : X \to Y \;:\Leftrightarrow\; f \subseteq X \times Y \land (\forall x \in X)(\exists! y \in Y)\big((x, y) \in f\big)$$

We use the notation $f : X \to Y$ to indicate that f is a function from X to Y.[3] The set X in this definition, called the *domain* of f, is the set of all inputs to f; we will use the notation $\mathfrak{dom}(f)$ to denote the domain of f. The set Y contains all of the *possible outputs* of f; we call this the *codomain* of f and use the notation $\mathfrak{cod}(f)$ to denote it. When f is a function from X to Y, we define the following convenient notation for every $x \in X$ and $y \in Y$.

$\mathfrak{dom}(\cdot)$ $\mathfrak{cod}(\cdot)$

$f(x) = y$

$$f(x) = y \;:\Leftrightarrow\; (x, y) \in f$$

[3] The expression $f : X \to Y$ is a *complete sentence* that is usually read in English as *"f is a function from X to Y."*

Here is a basic, but important, example of a function. Consider $\mathtt{id}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ given by $\mathtt{id}_{\mathbb{N}}(x) := x$ for every $x \in \mathbb{N}$. Below are some of the first few values of this function.

$$\begin{gathered}\mathtt{id}_{\mathbb{N}}(0) = 0 \\ \mathtt{id}_{\mathbb{N}}(1) = 1 \\ \mathtt{id}_{\mathbb{N}}(2) = 2 \\ \vdots\end{gathered}$$

identity function In fact, for any set A, we can define a version of this function—called the *identity function on A*—denoted by $\mathtt{id}_A$. We define $\mathtt{id}_A : A \to A$ such that $\mathtt{id}_A(a) := a$ for every $a \in A$.

In order for f to be a function from X to Y, every element $x \in X$ needs to have exactly one corresponding element $y \in Y$ such that $f(x) = y$. However, notice that the converse *does not have to hold!* Let's see why.

First, it is possible to have a function where two distinct inputs produce the same output. Formally, there might exist sets X and Y and a function $g : X \to Y$ such that:

$$(\exists x_1, x_2 \in X)\Big((x_1 \neq x_2) \land \big(g(x_1) = g(x_2)\big)\Big)$$

For example, consider the function $g : \{0, 1, 2, 3\} \to \{0, 1\}$ given by the mapping below.

$$\begin{aligned} g(0) &:= 0 \qquad & g(1) &:= 1 \\ g(2) &:= 0 \qquad & g(3) &:= 1 \end{aligned}$$

Second, it is possible for there to be elements of the codomain of a function that *are not actually outputs* of that function. Formally, there might exist sets X and Y and a function $h : X \to Y$ such that the following sentence is satisfied.

$$(\exists y \in Y)(\forall x \in X)\big(h(x) \neq y\big)$$

For example, consider the function $h : \mathbb{N} \to \mathbb{N}$ given by $h(x) := x + 1$ for each $x \in \mathbb{N}$. Notice that $0 \in \mathfrak{cod}(h)$, but $\big(\forall x \in \mathfrak{dom}(h)\big)\big(h(x) \neq 0\big)$. We will revisit these ideas soon.

7.2 Sequences and Iteration

Definition 7.3: Sequence.

sequence

Suppose that A is a set. We say that *f is an infinite sequence over A* $:\Leftrightarrow$ $f : \mathbb{N} \to A$. We say that *f is a finite sequence over A* $:\Leftrightarrow$ $(\exists n \in \mathbb{N})\big(f : n \to A\big)$. Usually, if we call something a *"sequence"* without qualification, it is assumed we are referring to an *infinite* sequence; though, of course, it is best to be specific when possible.[4]

[4]Note that we have ***not*** *yet defined* what *"infinity"* or *"finity"* are. We *do* ***not*** yet have a formal understanding of what it means for a set to be *"infinite"* or *"finite"* yet.

For example, the binary sequence "0, 1, 0, 1, 1, 0, 1, 1, 1, 0, ..." is formally just a function $\beta : \mathbb{N} \to \{0, 1\}$ where $\beta(0) := 0$, and $\beta(1) := 1$, and $\beta(2) := 0$, and $\beta(3) := 1$, *et cetera...*

Definition 7.4: Iterated Addition.

$\sum$

Let $f : \mathbb{N} \to \mathbb{N}$ be a sequence of natural numbers. We recursively define the *iterated sum* of finitely many values of f below for every $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

$$\begin{aligned}
\sum_{i=a}^{b} f(i) &:= 0 && \text{if } b < a \\
\sum_{i=a}^{a} f(i) &:= f(a) && \text{if } b = a \\
\sum_{i=a}^{\mathfrak{suc}(b)} f(i) &:= \left(\sum_{i=a}^{b} f(i)\right) + f\big(\mathfrak{suc}(b)\big) && \text{if } b > a
\end{aligned}$$

Theorem 7.1.

For all $n \in \mathbb{N}$, we have that $2 \cdot \sum_{i=0}^{n} i = n \cdot (n+1)$.

Proof. We will prove $(\forall n \in \mathbb{N})\Big(2 \cdot \sum_{i=0}^{n} i = n \cdot (n+1)\Big)$ by *weak induction.*

Basis Step:

Observe the following.

In the *basis step,* we need to prove $2 \cdot \sum_{i=0}^{0} i = 0 \cdot (0+1)$.

$$\begin{aligned}
2 \cdot \sum_{i=0}^{0} i &= 2 \cdot 0 && \text{by definition of iterated addition} \\
&= 0 && \text{by definition of multiplication} \\
&= 0 \cdot 0 && \text{by definition of multiplication} \\
&= (0 \cdot 0) + 0 && \text{by definition of addition} \\
&= 0 \cdot \mathfrak{suc}(0) && \text{by definition of multiplication} \\
&= 0 \cdot 1 && \text{by definition of 1} \\
&= 0 \cdot (0+1) && \text{because } (\forall x \in \mathbb{N})(0 + x = x)
\end{aligned}$$

Thus, we now know $2 \cdot \sum_{i=0}^{0} i = 0 \cdot (0+1)$ as desired.

Inductive Step:
Let $k \in \mathbb{N}$, and assume $2 \cdot \sum_{i=0}^{k} i = k \cdot (k+1)$. We can now observe the following.

In the *inductive step*, we must prove, for every $k \in \mathbb{N}$, $2\sum_{i=0}^{k} i = k(k+1)$ implies $2\sum_{i=0}^{\mathfrak{suc}(k)} i = \mathfrak{suc}(k)\big(\mathfrak{suc}(k)+1\big)$.

$$\begin{aligned}
2 \cdot \sum_{i=0}^{\mathfrak{suc}(k)} i &= 2 \cdot \left(\left(\sum_{i=0}^{k} i \right) + \mathfrak{suc}(k) \right) && \text{by definition of iterated addition} \\
&= \left(2 \cdot \sum_{i=0}^{k} i \right) + \big(2 \cdot \mathfrak{suc}(k)\big) && \text{by distributivity of } \cdot \text{ over } + \\
&= \big(k \cdot (k+1)\big) + \big(2 \cdot \mathfrak{suc}(k)\big) && \text{by the \textit{inductive hypothesis}} \\
&= \big(k \cdot (k+1)\big) + \big(2 \cdot (k+1)\big) && \text{because } (\forall x \in \mathbb{N})\big(\mathfrak{suc}(x) = x+1\big) \\
&= \big((k+1) \cdot k\big) + \big((k+1) \cdot 2\big) && \text{by commutativity of multiplication} \\
&= (k+1) \cdot (k+2) && \text{by distributivity of } \cdot \text{ over } + \\
&= (k+1) \cdot \big(k + (1+1)\big) && \text{because } 1+1=2 \\
&= (k+1) \cdot \big((k+1)+1\big) && \text{by associativity of addition} \\
&= \mathfrak{suc}(k) \cdot \big(\mathfrak{suc}(k)+1\big) && \text{because } (\forall x \in \mathbb{N})\big(\mathfrak{suc}(x) = x+1\big)
\end{aligned}$$

Thus, we have shown $2 \cdot \sum_{i=0}^{\mathfrak{suc}(k)} i = \mathfrak{suc}(k) \cdot \big(\mathfrak{suc}(k)+1\big)$ as required.

Therefore, we can conclude $(\forall n \in \mathbb{N})\Big(2 \cdot \sum_{i=0}^{n} i = n \cdot (n+1)\Big)$.

QED

Definition 7.5: Iterated Multiplication.

$\prod$ Let $f : \mathbb{N} \to \mathbb{N}$ be a sequence of natural numbers. We define the *iterated product* of finitely many values of f recursively below for every $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

$$\begin{aligned}
\prod_{i=a}^{b} f(i) &:= 1 && \text{if } b < a \\
\prod_{i=a}^{a} f(i) &:= f(a) && \text{if } b = a \\
\prod_{i=a}^{\mathfrak{suc}(b)} f(i) &:= \left(\prod_{i=a}^{b} f(i) \right) \cdot f\big(\mathfrak{suc}(b)\big) && \text{if } b > a
\end{aligned}$$