

## 5 Establishing a Foundation for Arithmetic

We recently showed that  $\cap\emptyset$  does not exist by proving that, if it did, then we could reproduce Russell's paradox within our theory of sets. The main problem with  $\cap\emptyset$  is that this set is, in a sense, *too big to exist*. The predicate  $\forall y(y \in \emptyset \Rightarrow z \in y)$  that defines what it means for a set  $z$  to be an element of  $\cap\emptyset$  is *true* for every set  $z$ . As a result, if  $\cap\emptyset$  were a set, we would be able to prove  $\forall x(x \in \cap\emptyset)$ .<sup>1</sup> Interestingly, this would imply that  $\cap\emptyset \in \cap\emptyset$ , so it would mean there is a set that contains itself as an element.

Notice that this property is not in-and-of-itself problematic. The existence of a set  $x$  such that  $x \in x$  is not *inherently contradictory*, and it does not pose the kind of existential problem that Russell's paradox does. In fact, the axioms that we have studied so far do nothing to *prevent* sets from containing themselves; it would not be possible to *disprove* the existence of a set  $x$  with the property  $x \in x$  just based on our current axiomatization.

It is at this point then that we must ask: should we allow sets to contain themselves? Does the concept of a "collection of things" either *entail* or *preclude* self-elementhood?

<sup>1</sup>Sets with this property are called *universal sets*. A consequence of our axiomatization of set theory is that *universal sets do not exist*.

### 5.1 The Axiom of Regularity

Most mathematicians, and certainly most reasonable people, would agree that a set that contains itself as an element would be a very strange object. If we think of sets as "containers" or "boxes," we certainly will never encounter a box in the real world that physically contains itself inside of itself. With this intuition, we should reject the existence of sets like  $x = \{x\}$ , and like  $a = \{b\}$  and  $b = \{a\}$ , and so on... But it turns out that there are a lot of complicated ways in which sets than be arranged so that they "eventually" contain themselves. What we would like is a *well-founded* set theory: one in which there are no infinitely long descending chains of  $\in$  elementhood.

**Axiom 6: Regularity.**

$$\forall x(x \neq \emptyset \Rightarrow \exists y(y \in x \wedge x \cap y = \emptyset)).^2$$

That is precisely what the strangely worded axiom above guarantees. Let's try to gain some insight into how this axiom works by proving the following theorem.

**Theorem 5.1.**

$$\forall x(x \notin x).$$

**Proof.** Let  $x$  be a set. Towards a contradiction, assume  $x \in x$ . By the *axiom of pairing*, we know the set  $p := \{x, x\}$  exists, and  $p = \{x\}$  by the *axiom of extensionality*. Observe that  $x \in \{x\}$  by definition, so that  $p \neq \emptyset$  by the *axiom of extensionality*.

Since  $p \neq \emptyset$ , the *axiom of regularity* tells us there exists  $z$  such that  $z \in p$  and  $p \cap z = \emptyset$ . From  $z \in p$ , we derive  $z \in \{x\}$  by the *axiom of extensionality*, so that  $z = x$  by definition. This means  $p \cap x = \emptyset$ . Now, notice  $x \in x$  by assumption, and  $x \in p$  because  $p = \{x\}$ , so that  $x \in p \cap x$ . This implies  $p \cap x \neq \emptyset$  because  $\forall z(z \notin \emptyset)$ .  $\blacksquare$  Therefore,  $x \notin x$ .

QED

With this axiom, we can actually prove much more than the simple that that sets can't contain *themselves*; we also can't have codependent cycles of sets that all mutually contain each other. The simplest example of this would involve two sets,  $x$  and  $y$ , such that  $x \in y$  and  $y \in x$ . We will now prove that no such sets exist.

<sup>2</sup>This is also known as the *axiom of foundation*.

**Theorem 5.2.**

$$\forall x \forall y (x \in y \Rightarrow y \notin x).$$

**Proof.** Let  $x$  and  $y$  be sets. Assume  $x \in y$ , and suppose, towards a contradiction, that  $y \in x$ . Consider the set  $p := \{x, y\}$ , which we know exists by the *axiom of pairing*. By the *axiom of regularity*, there exists some set  $z$  such that  $z \in p$  and  $p \cap z = \emptyset$ . Since  $z \in p$ , we know  $z \in \{x, y\}$  by definition, so that  $z = x$  or  $z = y$ . We now take two cases.

**Case 1:**

Suppose  $z = x$ , so that  $p \cap x = \emptyset$ . Recall  $y \in x$  by assumption. Notice  $y \in \{x, y\}$  by definition, implying that  $y \in p$  by the *axiom of extensionality*. This means  $y \in p \cap x$  by definition, which tells us  $p \cap x \neq \emptyset$ , contradicting  $p \cap x = \emptyset$ .  $\cancel{\square}$

**Case 2:**

Suppose  $z = y$ , so that  $p \cap y = \emptyset$ . Recall  $x \in y$  by assumption. Notice  $x \in \{x, y\}$  by definition, implying that  $x \in p$  by the *axiom of extensionality*. This means  $x \in p \cap y$  by definition, which tells us  $p \cap y \neq \emptyset$ , contradicting  $p \cap y = \emptyset$ .  $\cancel{\square}$

Therefore, we conclude that  $y \notin x$  as desired.

QED

## 5.2 The Axiom of Infinity

The time has finally come for us to confront our first axiom. We have been developing a theory of sets, and in the process have become acquainted with what it means to be a “set” as encoded by the axioms of Zermelo-Fraenkel set theory<sup>3</sup> with the exception of the *axiom of infinity*. We agreed that it *intuitively* asserts that “the set of all natural numbers exists.” To understand what this *means*, we need to recognize that the natural numbers are *arranged* in a very special way. The number 0 is the *first* natural number; the *next* natural number is 1; the *next* one after that is 2; and so on... Importantly, if we are searching for a particular natural number  $k$ , we can find  $k$  by *starting at 0* and then continually taking *discrete steps from one number to the next* until, eventually, we reach  $k$ .

Our notion of what a natural number “is” fundamentally relies on the observation that they can all be reached by *starting at zero* and then taking some finite number of *discrete steps* that move us *from one number to the next*. This means that we should first define who “zero” is and how to take these “steps.” We present these definitions below, without motivation, so that we can discuss their consequences.

**Definition 5.1: The Natural Number Zero.**

We define *the natural number zero* as  $0 := \emptyset$ .<sup>4</sup>

**Definition 5.2: The Successor Functional.**

Given an arbitrary set  $x$ , we define the *successor of  $x$*  to be  $\text{suc}(x) := x \cup \{x\}$ .

Formally speaking, everything that exists in set theory is a *set*, and therefore we are forced to define 0 as some kind of set. We chose to define 0 to be  $\emptyset$ , the empty set, for two reasons. First: the natural number 0 is supposed to be the “*first*” natural number; so, it is appropriate that we represent it by  $\emptyset$  since the empty set was the *first* specific set (that we understood) whose existence we formally proved. Second: we know that the empty set has no elements (*i.e.*, it is empty), so the quantity that 0 is supposed to represent corresponds—intuitively speaking—with the number of elements in  $\emptyset$ .

<sup>3</sup>Actually, this is a lie.

<sup>4</sup>We will eventually have other “*kinds*” of numbers than just the natural numbers, and they will also have a corresponding “zero” distinct from the natural number 0 we are defining here. For example, the “*integer zero*” and the “*rational number zero*” and the “*real number zero*,” if they were implemented as sets within set theory, would be *different* from the “*natural number zero*” because they would not be equal to  $\emptyset$ . In the interest of time, we will not be *constructing* these other number sets in this class.

0

suc(·)

Now that we know who “zero” is, the way we get to the “next” natural number is through the *successor functional* that we defined above: the natural number “one” is the successor of zero; the natural number “two” is the successor of one; and so on...

$$\begin{aligned}
 0 &:= \emptyset \\
 1 &:= \text{suc}(0) = \text{suc}(\emptyset) = \emptyset \cup \{\emptyset\} = \{\emptyset\} \\
 &= \{0\} \\
 2 &:= \text{suc}(1) = 1 \cup \{1\} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\} \\
 &= \{0, 1\} \\
 3 &:= \text{suc}(2) = 2 \cup \{2\} = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
 &= \{0, 1, 2\} \\
 &\vdots
 \end{aligned}$$

The effect that  $\text{suc}(x)$  has is to create a *new set with one additional element* thanks to the theorem  $\forall x(x \notin x)$ . Intuitively, then, we see that the natural number 1 is the set  $\{0\}$  containing *one* element; the natural number 2 is the set  $\{0, 1\}$  containing *two* elements; the natural number 3 is the set  $\{0, 1, 2\}$  containing *three* elements; and so on... Extending this pattern, we will see that the natural number  $n$  should correspond to  $\{0, 1, \dots, n - 1\}$ .

Taking these definitions of “zero” and “successor,” we now define what it means for a particular individual to be a *natural number* recursively as follows.

**Definition 5.3: Natural Number.**

We say that a given set  $n$  is a *natural number* precisely when any of the following are *true*.

1.  $n = 0$ .
2.  $n = \text{suc}(m)$ , where  $m$  is a *natural number*.

This definition implies that  $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$  is a natural number but  $\{\{\{\emptyset\}\}\}$  is not. We would like to put all of the things that are natural numbers, and only those things that are natural numbers, inside a set by themselves and claim that this set exists. This is what we want the *axiom of infinity* to assert. In order to contain all of the natural numbers—looking at our definition above of what a natural number is—we would need the set to have two properties:

1. The set must contain 0.
2. The set must contain successors for all of its elements.

The set of all natural numbers must *at least* have these two properties. Let’s give a name to sets with these two properties. We say that a set  $I$  is *inductive* if and only if, by definition, both  $0 \in I$  and  $\forall i(i \in I \Rightarrow \text{suc}(i) \in I)$  are satisfied. An inductive set would necessarily contain every natural number as an element, but it’s possible for a set to contain *more* than just natural numbers as elements and still qualify as an inductive set.<sup>5</sup> To prevent any “*excess*” elements, we will write the *axiom of infinity* as follows.

**Axiom o: Infinity.**

$\exists N(N \text{ is inductive} \wedge \forall I(I \text{ is inductive} \Rightarrow N \subseteq I))$ .

This axiom claims two things: an inductive set exists, and this inductive set is a subset of *every other* inductive set. Thanks to the *axiom of extensionality*, there can only be at most one set with the two properties described by the axiom above. We will call this unique set *the set of natural numbers* and use the symbol  $\mathbb{N}$  to denote it.

We obviously can not individually assign labels for *every* set that corresponds to a natural number; we ask that you continue this sequence of definitions so that every finite string of decimal digits corresponds to a set according to the pattern displayed to the left.

natural  
number

inductive

$\mathbb{N}$

<sup>5</sup>For example, a set that contains 0, all the successors of 0, and also  $\{\{\{\emptyset\}\}\}$ , and all of the successors of  $\{\{\{\emptyset\}\}\}$ , would qualify as inductive.