

4 A More Unified Theory of Sets

If we just stick to axioms 0, 1, 2, and 3, then *intuitively speaking* the only kinds of sets that we will ever be able to construct will contain at most two distinct elements.¹ This is an obviously unsatisfactory state of affairs. We will spend the rest of this *clavicula* discussing two new axioms that will allow us to construct a variety of sets.

4.1 The Power Axiom

The concept of a “subset” is fundamental to the study of sets. In fact, the *axiom schema of separation* asserts the existence of any particular subset of a given set x that we might want to talk about; all we need to do is specify a *wff* $\varphi(\cdot)$ and we can gather all of the elements z of x for which $\varphi(z)$ is *true* into their own set. Naturally, then, it would be nice to be able to gather *all of the subsets* of x into their own set, right? We would call such a set a *power set*, and it requires its own axiom to assert its existence.²

Definition 4.1: Power Set.

For any set x , we define the *power set of x* to be $\mathbb{P}(x) := \{z \mid z \subseteq x\}$.

Axiom 4: Power.

$\forall x \exists y (y = \{z \mid z \subseteq x\})$.³

Once we are ready, we will return to this definition and become intimately familiar with power sets. In the meantime, we will look at some of their basic properties below.

Lemma 4.1.

$\forall x (\emptyset \in \mathbb{P}(x))$.

Proof. Let x be an arbitrary set. Recall $\emptyset \subseteq x$ because $\forall y (\emptyset \subseteq y)$.⁴ By the definition of *set comprehension notation*, we have $\emptyset \in \{z \mid z \subseteq x\}$, implying $\emptyset \in \mathbb{P}(x)$ by definition.

QED

Lemma 4.2.

$\forall x (x \in \mathbb{P}(x))$.

Proof. Let x be an arbitrary set. Recall that $x \subseteq x$ because $\forall y (y \subseteq y)$.⁵ This implies that $x \in \{z \mid z \subseteq x\}$ by definition, so that $x \in \mathbb{P}(x)$ by definition.

QED

Corollary 4.1.

$\forall x (\mathbb{P}(x) \neq \emptyset)$.

Proof. Let x be a set. Recall that $x \in \mathbb{P}(x)$ as proven in the previous theorem. However, $x \notin \emptyset$ because $\forall y (y \notin \emptyset)$. Therefore, $\mathbb{P}(x) \neq \emptyset$ by the *axiom of extensionality*.

QED

Lemma 4.3.

$\forall x (x \subseteq \emptyset \Rightarrow x = \emptyset)$.

Proof. Let x be a set, and assume $x \subseteq \emptyset$. Let t be a set, and observe $t \notin \emptyset$ because $\forall z (z \notin \emptyset)$. Since $x \subseteq \emptyset$, this implies $t \notin x$,⁶ and thus $\forall y (y \notin x)$.⁷ Therefore, since *the empty set is unique*,⁸ we know $x = \emptyset$.

QED

¹Keep in mind that *we do not yet know how to count!* We know how to say that a set x contains *no elements* $\forall y (y \notin x)$ or *at least one element* $\exists y (y \in x)$. In fact, if we were clever enough, we could figure out how to say that a set contains any fixed finite number of distinct elements; however, since we don't know how to *compare the sizes of sets* yet, these statements don't carry much *formal weight* yet...

²Remarkably, even though the *axiom schema of separation* tells us that any definable subset of a given set x we care about exists, we can't use that axiom, nor any previous (or future) axioms, to prove that the *set containing all of the subsets of x* exists.

³If we use the notation introduced in the previous definition above, then we could rewrite this axiom as $\forall x \exists y (y = \mathbb{P}(x))$.

⁴Theorem 3.2 from *clavicula 3*.

⁵Lemma 3.1 from *clavicula 3*.

⁶This is an application of *universal elimination* followed by *modus tollens*.

⁷This is *universal introduction*.

⁸ $\forall a (\forall b (b \notin a) \Rightarrow a = \emptyset)$.

$\mathbb{P}(\cdot)$

Corollary 4.2.

$$\mathbb{P}(\emptyset) = \{\emptyset\}.$$

Proof. Observe that $\emptyset \subseteq \emptyset$ because $\forall x(\emptyset \subseteq x)$.⁹ This implies $\emptyset \in \mathbb{P}(\emptyset)$ by definition. Now, let t be a set and suppose $t \in \mathbb{P}(\emptyset)$. This means $t \subseteq \emptyset$ by definition. Now, by Lemma 4.3—which we just proved—we can see $t = \emptyset$. Thus, $\forall t(t \in \mathbb{P}(\emptyset) \Rightarrow t = \emptyset)$. The *axiom of extensionality* then lets us conclude $\mathbb{P}(\emptyset) = \{\emptyset\}$.

⁹Theorem 3.2 from *clavicula 3*.

QED

4.2 The Axiom of Union

Despite the power of the previous axiom, we are still far too limited in the kinds of new sets that we are allowed to form. We can't, for example, just merge two sets.

Definition 4.2: Union of Two Sets.

$x \cup y$ Given two sets x and y , we define the *union of x with y* to be $x \cup y := \{z \mid z \in x \vee z \in y\}$.

The union between two sets represents a way of “merging” them together by gathering all of their elements into one collection. For example, if $x := \{0, 1, 2\}$ and $y := \{1, 3, 4\}$, then $x \cup y = \{0, 1, 2, 3, 4\}$. It should be intuitively straightforward to understand why we would want $x \cup y$ to be a set when x and y are both sets. However, if we were to stop here and declare a new axiom along the lines of $\forall x \forall y \exists z(z = x \cup y)$, then we would be restricting ourselves to only being able to union *finitely many sets* together at a time.¹⁰

¹⁰Of course, we are being *intuitive* here; we don't formally know how to count yet and we certainly don't know what it means for a quantity to be “finite” (as opposed to “infinite,” for example).**Definition 4.3: Union Over a Set.**

$\cup x$ For any set x , we define the *union over x* to be $\cup x := \{z \mid \exists y(y \in x \wedge z \in y)\}$. This corresponds with iterating the binary “union” operation over all of the elements of x and accumulating the result. The alternative notation $\bigcup_{y \in x} y := \cup x$ is also common.¹¹

This definition given above provides a much more powerful notion of “unions” by describing the union of *all of the elements* of a set simultaneously, regardless of its “size.”

¹¹This is inspired by the common “summation” notation denoted with Σ ; we will encounter this notation later.**Axiom 5: Union.**

$$\forall x \exists y (y = \{w \mid \exists z(z \in x \wedge w \in z)\}).^{12}$$

If we only want to describe the union of, say, two sets $\{0, 1, 2\}$ and $\{1, 3, 4\}$, we could use our definition for $x \cup y$, but we could also see that $\cup\{x, y\}$ contains exactly the same elements. Thus, we can recover the existence of pairwise unions from this axiom.

¹²We could use our previous definition to rewrite this axiom to say $\forall x \exists y (y = \cup x)$.**Theorem 4.1: Existence of Pairwise Unions.**

$$\forall x \forall y \exists z (z = x \cup y).$$

Proof. Let x and y be sets. Observe that there exists a set p such that $\pi = \{x, y\}$ thanks to the *axiom of pairing*. Now, we know there exists a set μ such that $\mu = \cup\{x, y\}$ from the *axiom of union*. We will prove that $\mu = x \cup y$; to that end, let t be an arbitrary set.

$$\begin{aligned} t \in \mu &\Leftrightarrow t \in \cup\{x, y\} && \text{by definition of } \mu \\ &\Leftrightarrow t \in \{z \mid \exists w(w \in \{x, y\} \wedge z \in w)\} && \text{by definition of } \cup\{x, y\} \\ &\Leftrightarrow \exists w(w \in \{x, y\} \wedge t \in w) && \text{by definition of set compr.} \\ &\Leftrightarrow \exists w((w = x \vee w = y) \wedge t \in w) \\ &\Leftrightarrow \exists w((w = x \wedge t \in w) \vee (w = y \wedge t \in w)) \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \exists w(w = x \wedge t \in w) \vee \exists w(w = y \wedge t \in w) \\
&\Leftrightarrow t \in x \vee t \in y \\
&\Leftrightarrow t \in \{z \mid z \in x \vee z \in y\} && \text{by definition of set compr.} \\
&\Leftrightarrow t \in x \cup y && \text{by definition of } x \cup y
\end{aligned}$$

Therefore, by the *axiom of extensionality*, we can conclude that $\mu = x \cup y$.

QED

Definition 4.4: Intersection of Two Sets.

$x \cap y$ Given sets x and y , the *intersection of x with y* is given by $x \cap y := \{z \mid z \in x \wedge z \in y\}$.

Theorem 4.2: Existence of Pairwise Intersections.

$\forall x \forall y \exists z (z = x \cap y)$.

Proof. Let x and y be sets. We now know that $x \cup y$ exists thanks to the theorem we proved previously. By the *axiom schema of separation*, we know there exists a set θ such that $\theta = \{z \mid z \in x \cup y \wedge z \in x \cap y\}$. We will show $\theta = x \cap y$; to that end, let t be a set.

Suppose $t \in x \cap y$. Recalling $x \cap y \subseteq x \cup y$,¹³ we know $t \in x \cup y$. So, $t \in \theta$ by definition. Suppose $t \in \theta$. Then, $t \in x \cup y \wedge t \in x \cap y$ by definition, implying $t \in x \cap y$.

¹³Problem 1.d. on problem set 3.

Therefore, by the *axiom of extensionality*, we can conclude that $\theta = x \cap y$.

QED

In the same way that we defined the *union over a set* in order to have a more powerful notion of “unions,” we can define what it means to take the “intersection over a set.”

Definition 4.5: Intersection Over a Set.

$\cap x$ For any set x , we define the *intersection over x* to be $\cap x := \{z \mid \forall y (y \in x \Rightarrow z \in y)\}$. This corresponds with iterating the binary “intersection” operation over all of the elements of x and accumulating the result. An alternative notation that is sometimes encountered is $\bigcap_{y \in x} y := \cap x$.

Following our intuition from before, we might be tempted to say that “arbitrary intersections over sets should exist” in the same way we said arbitrary unions over sets should exist to justify the *axiom of union*. Before we go that far, let’s take a moment to think about the consequences this would have, and the consequences that our most recent axiom has already had. We know that, for *any* set x , the object denoted by $\cup x$ exists. In particular, we should try to understand what this means if this knowledge is applied to our *favorite set*: the empty set. What is $\cup \emptyset$?

Lemma 4.4.

$\cup \emptyset = \emptyset$.

Proof. Towards a contradiction, suppose there exists a set x such that $x \in \cup \emptyset$. Now, recalling $\cup \emptyset = \{z \mid \exists y (y \in \emptyset \wedge z \in y)\}$, this means $\exists y (y \in \emptyset \wedge x \in y)$ by definition. Thus, there exists a set w such that $w \in \emptyset$ and $x \in w$. However, we know $w \notin \emptyset$ because $\forall y (y \notin \emptyset)$. \nexists Therefore, we conclude that $\cup \emptyset = \emptyset$.

QED

Having satisfied ourselves with unions, we return now to our concern regarding “arbitrary intersections over sets” and ask ourselves the analogous question: what is $\cap \emptyset$? As it turns out, the collection referred to by $\cap \emptyset$ *does not exist!*

Lemma 4.5.

$$\forall x(x \neq \cap \emptyset).$$

Proof. Towards a contradiction, suppose that there exists a set U such that $U = \cap \emptyset$. By the *axiom schema of separation*, there exists a set R such that $R = \{z \mid z \in U \wedge z \notin z\}$. Now, either $R \in R$ or $R \notin R$, so we will take two cases.

Case 1:

If $R \in R$, then $R \in U$ and $R \notin R$ by definition, contradicting the premise $R \in R$. ζ

Case 2:

Suppose $R \notin R$. Then, by definition, $\neg(R \in U \wedge R \notin R)$, which is equivalent to $R \notin U \vee R \in R$. Recall $\forall y(y \notin \emptyset)$ and observe the following logical deduction.

$$\forall y(y \notin \emptyset) \Rightarrow \forall y(y \notin \emptyset \vee R \in y) \Rightarrow \forall y(y \in \emptyset \Rightarrow R \in y)$$

This then implies $R \in \{z \mid \forall y(y \in \emptyset \Rightarrow z \in y)\}$ by definition, so that $R \in \cap \emptyset$ by definition, and thus $R \in U$ by the *axiom of extensionality*. In light of our prior realization $R \notin U \vee R \in R$, we see $R \in R$,¹⁴ contradicting our assumption $R \notin R$. ζ

¹⁴This is an application of the *disjunctive syllogism*.

We must therefore conclude that there *does not exist* any set equal to $\cap \emptyset$.

QED

As a result, we can not a complementary axiom to the *axiom of union* that would allow for the existence of $\cap x$ for any set x . If we did, we could recreate the argument above, reproduce Russell's paradox, and introduce a contradiction into our theory. However... as it turns out... we can prove that $\cap x$ *does* in fact exist as long as the set x is *nonempty*! This is an immensely important result that we will soon come to appreciate when we revisit the *axiom of infinity*.

Theorem 4.3: Existence of Nonempty Intersections.

$$\forall x(x \neq \emptyset \Rightarrow \exists y(y = \cap x)).$$

Proof. Let x be a set, and suppose $x \neq \emptyset$. The *axiom schema of separation* states that there exists a set Ω such that $\Omega = \{z \mid z \in \cup x \wedge z \in \cap x\}$. We'll show $\Omega = \cap x$. Let t be a set.

First, suppose that $t \in \Omega$, so that $t \in \{z \mid z \in \cup x \wedge z \in \cap x\}$ by definition, telling us that $t \in \cup x$ and $t \in \cap x$ by definition. We can clearly see $t \in \cap x$ as required.

Conversely, suppose $t \in \cap x$. By definition, this means $\forall y(y \in x \Rightarrow t \in y)$. Now, we are compelled to recall $x \neq \emptyset$, which means there exists a set w such that $w \in x$ by the *axiom of extensionality*. Using our earlier observation $\forall y(y \in x \Rightarrow t \in y)$, this then implies $t \in w$. Thus, $\exists y(y \in x \wedge t \in y)$, which means $t \in \cup x$ by definition. Therefore, since we know $t \in \cup x$ and $t \in \cap x$, we obtain $t \in \Omega$ by definition.

We now invoke the *axiom of extensionality* to conclude $\Omega = \cap x$.

QED