3 Expanding our Notion of Set

Thanks to the *axiom of extensionality,* we now know that sets are equal when they have exactly the same elements. Thanks to the *axiom of pairing,* we now also know that we can take two existing sets x and y and construct the unordered pair $\{x, y\}$ with the knowledge that $\{x, y\}$ also exists.

In the previous Clavicula, we defined what it means for one set x to be a *subset* of another set y. We remind you of the definition, for two arbitrary sets x and y, below.

$$x \subseteq y \quad :\Leftrightarrow \quad \forall z\big(z \in x \;\; \Rightarrow \;\; z \in y\big)$$

This *"relation"*[1] we've defined has several important properties. Three of these properties show us that $\subseteq$ is an example of a *partial order* (at least *in spirit*).

[1] This is not a "relation" *technically speaking* if you have encountered that term in other contexts, but it is *"spiritually"* a relation.

Lemma 3.1.
The following three statements hold.

1. $\forall x(x \subseteq x)$. *Reflexivity*
2. $\forall x \forall y\big((x \subseteq y \wedge y \subseteq x) \;\Rightarrow\; x = y\big)$. *Antisymmetry*
3. $\forall x \forall y \forall z\big((x \subseteq y \wedge y \subseteq z) \;\Rightarrow\; x \subseteq z\big)$. *Transitivity*

We will come to understand these properties more intimately later.

We also defined some notation, in the previous Clavicula, for referring to the *"collection of things that satisfy a given property."* Given a well-formed formula φ containing exactly one free variable, we use the notation $\{x \mid \varphi(x)\}$ to denote the collection of all things that satisfy φ. Reflecting on these definitions and axioms, one may be lead to consider whether or not an "empty set" exists. This is a remarkably important question.

First, we should define *what* this empty set would be; once we know *what* we're talking about, we can think about whether or not we can prove it exists. We will define the[2] *empty set*,[3] denoted by the symbol $\varnothing$, as follows.

[2] It is interesting that we should use the word *"the"* to describe this thing…

[3] It is somewhat inappropriate for us to be calling this the *empty* ***"set"*** since we do not yet *know* that this *thing* ***exists***. However, we will soon resolve this problem by *proving* that the thing that $\{x \mid x \neq x\}$ denotes exists.

Definition 3.1: Empty Set.
$\varnothing$ $\varnothing := \{x \mid x \neq x\}$.

Before knowing that this *"thing"* exists, we can already start proving it has some very nice properties—properties would should *expect* it to have based on its name.[4] Most importantly, we should verify that this "empty set" is actually *empty.* If we think about *empty* what it means for an arbitrary set x to be *empty*, it should mean $\forall y(y \notin x)$. We prove that $\varnothing$ has this property below.

[4] If we did not have an actual existence proof for this object, then a lot of what we're about to do would be a waste of time. However, since we *will* actually have an existence proof for $\varnothing$ later, we do not toil in vain.

Theorem 3.1: The Empty Set is Empty.
$\forall x(x \notin \varnothing)$.

Proof. Let x be an arbitrary set and suppose, towards a contradiction, that $x \in \varnothing$, which literally means $x \in \{z \mid z \neq z\}$ by definition. Then, we know that $x \neq x$ by definition. However, this contradicts the fact that $x = x$. ↯ Therefore, $x \notin \varnothing$.

QED

The empty set has an interesting relation with the $\subseteq$ symbol we defined earlier. In fact, in a certain precise sense, the empty set is *"very nice"* and exhibits an important *universal property.* We hint at this *universal property* below by providing three proofs of the fact that the empty set is a subset of any set.

Theorem 3.2.
$\forall x(\varnothing \subseteq x)$.

Proof 1. Let x be a set. Let z be an arbitrary set. Since $\forall w(w \notin \varnothing)$, we know $z \notin \varnothing$. Thus, we can say $z \notin \varnothing \lor z \in x$. This is equivalent to $z \in \varnothing \Rightarrow z \in x$.

Therefore, since z was an arbitrary set, we have shown that $\forall w(w \in \varnothing \Rightarrow w \in x)$. This means precisely that $\varnothing \subseteq x$ by definition.

QED

Proof 2. Let x be a set. Let z be a set and assume $z \in \varnothing$. Recall that $\forall w(w \notin \varnothing)$, so that $z \notin \varnothing$. Hence, by the *principle of explosion,* we have $z \in x$.

We have now shown that $\forall w(w \in \varnothing \Rightarrow w \in x)$, meaning $\varnothing \subseteq x$ by definition.

QED

Proof 3. Let x be a set. Let z be a set and assume $z \in \varnothing$. Towards a contradiction, assume $z \notin x$. Recalling $\forall w(w \notin \varnothing)$, we know that $z \notin \varnothing$. However, this contradicts our assumption that $z \in \varnothing$. ↯ Therefore, we must have $z \in x$.
So, we have shown $\forall w(w \in \varnothing \Rightarrow w \in x)$. We therefore conclude $\varnothing \subseteq x$ by definition.

QED

Finally, we should justify our use of the word *"the"* when referring to *the* empty set. We will provide two proofs that any set that happens to be empty *is* actually equal to $\varnothing$.

Theorem 3.3: The Empty Set is Unique.
$\forall x\big(\forall y(y \notin x) \Rightarrow x = \varnothing\big)$.

Proof 1. Let x be a set and assume that $\forall y(y \notin x)$. Towards a contradiction, suppose $x \neq \varnothing$. By the *axiom of extensionality,* there then exists some z such that $z \in x \land z \notin \varnothing$ or $z \notin x \land z \in \varnothing$. We now take two cases based on this disjunction.[5]

Case 1:
Suppose $z \in x$ and $z \notin \varnothing$. Recall $\forall y(y \notin x)$, implying $z \notin x$, contradicting $z \in x$. ↯

Case 2:
Suppose $z \notin x$ and $z \in \varnothing$. Recall $\forall w(w \notin \varnothing)$, so that $z \notin \varnothing$, contradicting $z \in \varnothing$. ↯

Since we have derived contradictions in both cases, our initial assumption must have been mistaken, and we must therefore conclude that $x = \varnothing$.

QED

[5]Note that what we are doing here, typically called a *proof by cases*, is just an example of applying the *disjunction elimination* theorem that was proven in *Problem Set 2*. As a reminder, *disjunction elimination* states the following: $\alpha \lor \beta,\ \alpha \to \delta,\ \beta \to \delta \vdash \delta$

Proof 2. Let x be a set. We will prove the contrapositive of the claim.[6] With this goal in mind, assume $x \neq \varnothing$. By the *axiom of extensionality,* this means that there exists some z such that $(z \in x \land z \notin \varnothing) \lor (z \notin x \land z \in \varnothing)$. However, if we recall $\forall w(w \notin \varnothing)$, then we can see $z \notin \varnothing$, thereby refuting $z \notin x \land z \in \varnothing$. From these results, we can then deduce $z \in x \land z \notin \varnothing$,[7] so that $z \in x$. We have therefore shown $\exists y\big(y \in x\big)$.

We conclude by noticing $(x \neq \varnothing) \Rightarrow \exists y\big(y \in x\big)$ is equivalent to $\forall y\big(y \notin x\big) \Rightarrow (x = \varnothing)$.

QED

[6]Recall that the *contrapositive* of a given conditional statement $\alpha \to \beta$ is an *equivalent* conditional statement that looks like $\neg\beta \to \neg\alpha$.

[7]This is an example usage of the *disjunctive syllogism* from *Problem Set 2,* which says: $\alpha \lor \beta,\ \neg\beta \vdash \alpha$

Now that we have gotten to know $\varnothing$ a little better, we should start to become earnestly worried about the question of its existence. If it turns out that we can not prove that $\varnothing$ exists, then writing proofs about its supposed properties becomes much less interesting. Resolving this question requires us to look more deeply at the $\subseteq$ relation.

3.1 *The Axiom Schema of Separation*

In the same way we motivated axioms 1 and 2 based on our intuitive notion of what a set *should* be like, and what kinds of things we *should* be *allowed* to do to construct sets, we might naturally be lead to wonder whether *any* ***definable*** *collection of things* should be considered a set. At first glance, this should sound completely reasonable; we even have special notation—*set comprehension notation*—for describing "the collection of all objects that have a property." If we can precisely express a property through a well-formed formula φ, then why shouldn't we be able to talk about *"the set* $\{x \mid \varphi(x)\}$ *of all things that have the property expressed by* φ*"?* It sounds like we are about to introduce a new axiom that says something like:

For any well-formed formula φ with one free variable, $\exists x\big(x = \{z \mid \varphi(z)\}\big)$.

Conveniently, this would also neatly resolve the question of the exists of the empty set. Recall that $\varnothing = \{x \mid x \neq x\}$ by definition; if we notice $x \neq x$ is equivalent to $\neg(x = x)$, which is a well-formed formula with one free variable, then this proposed axiom would immediately tell us that $\varnothing$ exists. But… what are the consequences for our hubris…?

When thinking through the ramifications, we could consider sets like $\{x \mid x = x\}$ and $\{x \mid x \in x\}$, what properties they would have, and whether their existence would be unexpected in the theory of sets we are developing.[8] Eventually, after enough time and reflection, we might arrive at at a collection like $\mathfrak{R} := \{x \mid x \notin x\}$. If we take our new proposed axiom at face-value, then we should be able to say $\mathfrak{R}$ exists, and we should therefore expect to be able to ask questions about $\mathfrak{R}$—the same kinds of questions that we would ask about any other set—and expect to get coherent answers. To be more precise, we should be able to use $\mathfrak{R}$ as a term in a sentence[9] that expresses a truth value. For example, we should be able to say either $\mathfrak{R} \in \mathfrak{R}$ or $\mathfrak{R} \notin \mathfrak{R}$. So… which is it?

[8] There *are* serious problems that would result from considering those *things* to be *"sets,"* but we are not ready to discuss that yet.

[9] Recall that a *sentence* is, formally, a well-formed formula with no free variables.

Well, if we suppose $\mathfrak{R} \in \mathfrak{R}$, then that means $\mathfrak{R} \in \{x \mid x \notin x\}$ by the definition of $\mathfrak{R}$. This implies $\mathfrak{R} \notin \mathfrak{R}$ by the definition of set comprehension notation. But that contradicts our assertion that $\mathfrak{R} \in \mathfrak{R}$… $\lightning$

Okay, what if we suppose $\mathfrak{R} \notin \mathfrak{R}$? This is equivalent to $\neg\big(\mathfrak{R} \in \{x \mid x \notin x\}\big)$ by the definition of $\mathfrak{R}$, which means $\neg(\mathfrak{R} \notin \mathfrak{R})$ by the definition of set comprehension notation. But that just says $\mathfrak{R} \in \mathfrak{R}$,[10] clearly contradicting our earlier claim that $\mathfrak{R} \notin \mathfrak{R}$… $\lightning$

[10] … by *double negation*…

However, we know that $\mathfrak{R} \in \mathfrak{R} \vee \mathfrak{R} \notin \mathfrak{R}$ because that's a *theorem* of the first-order logic. If we combine this theorem with the arguments in the preceding paragraphs, and if we remember the theorem named *disjunction elimination,* then we will be able to *prove a contradiction,* which (by the *principle of explosion*) will let us *prove whatever we want!* This makes our "theory" completely meaningless! We can not allow collections like $\mathfrak{R}$ to exist; the mere existence of such objects introduces contradictions!

Now, let's take a step back and recall the $\subseteq$ relation we introduced earlier. Instead of asserting the existence of the collection of *all* objects that have some property, what if we collect only those objects *that already belong to an existing set* that have the desired property? Instead of asserting that $\{x \mid \varphi(x)\}$ always exists, why don't we instead make the more modest assertion that $\{x \mid x \in A \wedge \varphi(x)\}$ exists whenever A exists? This would be the *subset* of A consisting of all those elements of A that have the property φ. As it turns out, this fixes everything waow.

Axiom 3: Schema of Separation.
For any *wff* φ with one free variable, we take the following sentence on faith.
$\forall x \exists y \big(y = \{ z \mid z \in x \wedge \varphi(z) \} \big)$.

This axiom allows us to know that *arbitrary subsets* of sets that exist also exist.

3.1.1 *Some More Notation*

In light of our new axiom, we will introduce some new notation that will make communication involving more complex sets easier. For any well-formed formula φ containing one free variable, and for any set x, we define the following new set notation.

$$\{ z \in x \mid \varphi(z) \} := \{ z \mid z \in x \wedge \varphi(z) \}$$

Using this notation, we could rephrase the *axiom schema of separation*, for a given *wff* φ, as the statement $\forall x \exists y \big(y = \{ z \in x \mid \varphi(z) \} \big)$ since this is equivalent to the sentence $\forall x \exists y \big(y = \{ z \mid z \in x \wedge \varphi(z) \} \big)$. We introduce two more useful notations for any set x.

$$\begin{aligned} (\forall y \in x)\big(\varphi(y)\big) \quad &:\Leftrightarrow \quad \forall y \big(y \in x \ \Rightarrow \ \varphi(y) \big) \\ (\exists y \in x)\big(\varphi(y)\big) \quad &:\Leftrightarrow \quad \exists y \big(y \in x \wedge \varphi(y) \big) \end{aligned}$$

This notation will be very useful in the future,[11] but it is important to keep in mind that $(\forall y \in x)$ ***is not a sentence*** *on its own!* The expression $(\exists y \in x)$ ***does not mean anything****!* These fragments ***must*** be attached to a *"body"* to create a well-formed sentence.

[11] … and, importantly, while this notation appears in many locations in other books, notes, and resources, there are often *subtle variations* on this notation (for example, dropping parentheses, or using other forms of punctuation) that *mean* the same thing…