1 Important Theorems of the First-Order Logic

Now that we understand what *predicates, terms,* and *quantifiers* are, it is time for us to define the kinds of expressions we are interested in communicating in the first-order logic.[1] We say λ is a *well-formed formula* if and only if any of the following are satisfied.

1. $\lambda = \top$ or $\lambda = \bot$.
2. $\lambda = \varphi(x_1, \ldots, x_n)$ where φ is an n-ary predicate, and $x_1, \ldots, x_n$ are terms.[2]
3. $\lambda = \neg(\varphi)$ where φ is a *well-formed formula.*
4. $\lambda = (\varphi) \wedge (\psi)$ where φ and ψ are *well-formed formulae.*
5. $\lambda = (\varphi) \vee (\psi)$ where φ and ψ are *well-formed formulae.*
6. $\lambda = (\varphi) \rightarrow (\psi)$ where φ and ψ are *well-formed formulae.*
7. $\lambda = (\varphi) \leftrightarrow (\psi)$ where φ and ψ are *well-formed formulae.*
8. $\lambda = \forall x(\varphi)$ where φ is a *well-formed formula.*
9. $\lambda = \exists x(\varphi)$ where φ is a *well-formed formula.*

[1] Just like when we defined what a proposition was, this will be a *recursive* definition.

[2] Here, we intuitively take n to be a positive integer.

A well-formed formula λ is called a *sentence* when λ contains no free variables.[3] In addition to our four quantified rules of inference, there are several important theorems regarding sentences in the first-order logic that we should keep in mind. We summarize these theorems here, in part, because there are some ***typos page 31*** of the lecture notes.

[3] This implies every term that appears in λ is either a *constant* or a *variable bound by a quantifier.*

Theorem 1.1: Negation of Quantifiers.
For any well-formed formula φ with one free variable, the following equivalences hold.

$$\neg\forall x(\varphi(x)) \equiv \exists x(\neg\varphi(x))$$
$$\neg\exists x(\varphi(x)) \equiv \forall x(\neg\varphi(x))$$

Theorem 1.2: Quantifier Shift.
For any well-formed formula φ containing two free variables, the following hold.

$$\forall x\forall y\big(\varphi(x,y)\big) \equiv \forall y\forall x\big(\varphi(x,y)\big)$$
$$\exists x\exists y\big(\varphi(x,y)\big) \equiv \exists y\exists x\big(\varphi(x,y)\big)$$

This means we can commute quantifiers *when they are the same;* however, universal and existential quantifiers *do **not*** necessarily commute with *each other!*

$$\exists x\forall y\big(\varphi(x,y)\big) \;\vdash\; \forall y\exists x\big(\varphi(x,y)\big)$$
$$\forall x\exists y\big(\varphi(x,y)\big) \;\nvdash\; \exists y\forall x\big(\varphi(x,y)\big)$$

Theorem 1.3: Distribution of Quantifiers.
For any well-formed formulae[4] φ and ψ we can *distribute quantifiers* as shown below.

[4] … each containing exactly one free variable…

$$\forall x(\varphi(x) \wedge \psi(x)) \equiv \forall x(\varphi(x)) \wedge \forall x(\psi(x))$$
$$\exists x(\varphi(x) \vee \psi(x)) \equiv \exists x(\varphi(x)) \vee \exists x(\psi(x))$$

This means *universal quantifiers* can be distributed *over conjunctions,* and *existential quantifiers* can be distributed over *disjunctions.* However, the opposite *does **not** hold!*

$$\forall x(\varphi(x) \vee \psi(x)) \;\nvdash\; \forall x(\varphi(x)) \vee \forall x(\psi(x))$$
$$\exists x(\varphi(x)) \wedge \exists x(\psi(x)) \;\nvdash\; \exists x(\varphi(x) \wedge \psi(x))$$