Terminology in the First-Order Logic

The vocabulary surrounding the first-order logic can sometimes be confusing; we attempt to clarify some of this vocabulary here.

We have talked about *propositions* in one of two ways. Intuitively, a proposition is a declarative statement[1] that is either *true* or *false* but not both at the same time. This informs our *semantics* for the propositional logic. To begin defining the *syntax* of the propositional logic, we formally—recursively—define a string of symbols λ to be a proposition precisely when any of the following conditions hold.

[1] *i.e.,* an assertion, a claim

1. $\lambda = \top$
2. $\lambda = \bot$
3. $\lambda = \neg(\varphi)$ for some proposition φ
4. $\lambda = (\varphi) \wedge (\psi)$ for some propositions φ and ψ
5. $\lambda = (\varphi) \vee (\psi)$ for some propositions φ and ψ
6. $\lambda = (\varphi) \to (\psi)$ for some propositions φ and ψ
7. $\lambda = (\varphi) \leftrightarrow (\psi)$ for some propositions φ and ψ

In the zeroth-order logic, we were mainly concerned with expressing and analyzing properties of *"propositions"* because they are the *carriers of truth values.* This will not change: in the first-order logic, we are still interested in expressing sentences that have *one and only one consistent truth value.* However, the zeroth-order logic doesn't have the ability to talk *about objects;* propositions don't have any "internal structure," so all you can know about a proposition is that it is either *true* or *false* but not both. You can't actually see what any proposition *says* in the zeroth-order logic.

We would like to augment the zeroth-order logic with a semantics and syntax that will allow us to *talk about objects.* This requires we set up a few (mostly informal) definitions.

In order to start talking about objects, we first establish what kind of objects we're interested in talking about. This—the collection of all objects that are *"under discussion"* for our purposes—is called a *universe of discourse*.

A *term* is something that *refers to an object* in our universe of discourse. A term that refers to a *specific or particular object* is called a *constant*. For example, 7 is a constant symbol (if we have a universe of discourse like "the collection of all numbers"). If we define $x := 7$, then x would also be a constant symbol. We could also say something like the following.

"Let n be a natural number."

This defines a symbol n that refers to a "natural number." The natural number that n refers to is *arbitrary* in the sense that, based on just that sentence above, n could refer to *any* of the numbers $0, 1, 2, 3, \ldots$ However, n can only refer to *one* of them… because n is a *number*. The symbol n can't refer to two different numbers at the same time; otherwise, we would be in a situation where $n = 6$ and also $n = 7$, and we would then be able to prove $6 = 7$. ↯ The point of this discussion is that this phrase introduced a ***constant symbol***, not a *variable.*

A *variable* is a term that refers to a *generic object* as opposed to any particular object. In order to make sense of a variable, it ***must be quantified within some scope***, making it a *bound variable*. A variable that is not bound by a quantifier is called a *free variable*.

Let's look at the following example.

$$\text{Let } w := 7.$$
$$\forall x\Big(\big(\varphi(x) \wedge \psi(w,y)\big) \to \exists z\big(\rho(x,z)\big)\Big).$$

The sentence above involves four terms: w, x, y, and z. The term w is a *constant* since it refers to the object 7. The *scope* of this definition—the region where we understand the symbol w to be referring to the object 7—begins at that "Let …" sentence, and ends when we stop talking.[2]

[2]For example, in the context of a proof, the scope would end at the end of the proof.

The other three terms are *variables*. The variables x and z are both bound by quantifiers.

This sentence has the form $\forall x(\ldots)$. The $\forall$ symbol is immediately followed by x; this syntax denotes that x is bound by $\forall$, and the *scope of quantification* is indicated by the parentheses.[3] Within these parentheses, whenever we see the symbol x, it refers to some object from our universe of discourse. The fact that x was quantified universally by $\forall$ means that the statement *inside the parentheses* is being asserted *for every object* in our universe of discourse, and our variable x gives us a way of talking about those objects so that we can describe them using predicates.

[3]The *scope* of a variable bound by a quantifier is the syntactic region where the variable *has any meaning*.

The variable z is also bound by a quantifier—in this case, the existential $\exists$ quantifier. That part of the sentence says $\exists z\big(\rho(x,z)\big)$. The portion $\rho(x,z)$ is the only part of this sentence where z is quantified; that is the *scope* of z. Because this part of the sentence is contained within the parentheses that defined the scope of x, we know that the symbol x here refers to the x that was bound by the universal $\forall$ quantifier. It is *the same* x that appears in the $\varphi(x)$ portion of the sentence. Those two x must refer to the same object (since the same symbol is being used to refer to them); they are the *same term*.

The symbol y is ***not a constant*** because we were never introduced to who y refers to by, for example, a definition. So, y must be a variable; those are the only kinds of terms we have. However, y is ***not bound*** by any quantifier. That means y is a ***free variable*** in this sentence.

Sentences with *free variables* ***do not mean anything***! The only kinds of statements that express meaning—that carry *truth values*—are sentences with ***no free variables***.