Problem Set 1

Discrete Structures

Due on the ***7th day of September of the year of our Lord 2026*** *at 11:59 pm*

Justify each of your answers with an argument. An unjustified answer is worse than incorrect.

Consider the binary predicate introduced below for the purposes of this problem set.

$$=(x, y) := \text{"The object referred to by } x \text{ is the same as the object referred to by } y\text{."}$$

We assume the following three axioms about equality are satisfied in any universe.

$$\forall x(x = x)$$
$$\forall x \forall y\big((x = y) \to (y = x)\big)$$
$$\forall x \forall y \forall z\Big(\big((x = y) \land (y = z)\big) \to (x = z)\Big)$$

Note that, since we do not yet know the rules of inference for the first-order logic, we can not formally prove anything yet. Nonetheless, all answers should be justified by an informal argument that is as precise and rigorous as possible.

1. a. Construct a proposition that is logically equivalent to $\top$ if and only if the universe of discourse contains no objects.

   b. Construct a proposition that is logically equivalent to $\top$ if and only if the universe of discourse contains exactly one object.

   c. Construct a proposition that is logically equivalent to $\top$ if and only if the universe of discourse contains exactly two objects.

2. According to our recursive definition of a well-formed formula, a proposition can only be written using $\neg$, $\land$, $\lor$, $\to$, and $\leftrightarrow$ as logical connectives. Observe that, for any propositions p and q, we have the following logical equivalences.[1]

$$p \to q \equiv \neg p \lor q$$
$$p \land q \equiv \neg\big((\neg p) \lor (\neg q)\big)$$

   Does there exist a single binary logical connective such that any proposition can be rewritten as an equivalent expression involving only that logical connective?

3. Consider the infinite sequence of sentences indexed by natural numbers

$$\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_i, \dots$$

   where each sentence σ_i asserts that each of the subsequent sentences are *false*.[2]

$$\sigma_i := \text{"}\sigma_j \text{ is false for every natural number } j \text{ greater than } i\text{."}$$

   By fixing a universe of discourse containing all of the natural numbers, we can rephrase the definition of σ_i equivalently as "$\forall j\big((j > i) \to (\neg\sigma_j)\big)$." For example, the sentence σ_3 is equivalent to "$\forall j\big((j > 3) \to (\neg\sigma_j)\big)$," which is the assertion that the sentences σ_4, σ_5, σ_6, … and so on are all *false*.

   What are the truth values, if any, of the sentences in this sequence?[3][4]

[1] There are several ways we can justify these equivalences even if we don't know how to *prove* them yet. For example, we could argue that the truth tables for $\neg\neg p$ and p are identical, as they are for $p \to q$ and $\neg p \lor q$, and for $p \land q$ and $\neg((\neg p) \lor (\neg q))$.

[2] The variable i in the definition of σ_i ranges over all of the natural numbers $0, 1, 2, \dots$

[3] Your argument should describe the truth value, if any, of *every* sentence in this sequence.

[4] HINT: Try first to determine the truth value of σ_0. Once you have worked this out, think about how your argument might apply to an arbitrary sentence σ_i in the sequence.