Solution Set 0

Discrete Structures

31st day of August of the year of our Lord 2026

Justify each of your answers with an argument. An unjustified answer is worse than incorrect.

1. We will determine the truth value of each of the following sentences.

   a. "Every integer is either even or odd." is *true*.

   We define a number z to be *"even"* if and only if z is an integer multiple of 2. Similarly, we will say that z is *"odd"* if it is one more than a multiple of 2.[1]

   With these definitions and some intuition, we can see that 0 is even because 0 is an integer and $0 = 0 \cdot 2$. Now, given an arbitrary even integer x, we know that $x = 2z$ for some integer z according to our definition. We then have

   $$x = 2z = 2\big((z-1)+1\big) = 2(z-1)+2,$$

   which implies x can be expressed as two more than another even integer. We should then be able to obtain any positive even integer we want by starting at 0 and adding 2 enough times to reach that integer.[2] By similar reasoning, we can obtain every negative even integer by starting at 0 and adding -2 repeatedly. This accounts for all of the even integers, and the only integers left over must be strictly between the ones we've just accounted for. In other words, the non-even integers must appear as clusters between the even integers.

   Notice, however, that if we let x_1 and x_2 be even integers, so that $x_1 = 2z_1$ and $x_2 = 2z_2$ for some integers z_1 and z_2, then $x_1 - x_2 = 2z_1 - 2z_2 = 2(z_1 - z_2)$, so that the difference between z_1 and z_2 is even. This prevents $z_1 - z_2 = 1$, so that no two even integers are consecutive.[3] Thus, there is *at least one* integer that lies between any two *consecutive* even integers.[4] Since adding two to an arbitrary even integer also produces an even integer,[5] there is *at most one* integer lying between any two *consecutive* even integers. It then follows that there is *exactly one* integer between any two even integers.

   Now, given an arbitrary integer $2z_0$, we can see that $2z_0 < 2z_0 + 1 < 2z_0 + 2$, and the even integers $2z_0$ and $2z_0 + 2$ are consecutive. According to our definition, $2z_0 + 1$ is clearly odd beacuse $2z_0$ is even. Since this applies to any even integer, we see that all of the integers between consecutive even integers must be odd. We therefore conclude that every integer must be even or must be odd.

   b. "Every non-empty set of integers has a least element." is *false*.

   To see that this sentence is *false*, we will find an example of a non-empty set of integers with no least element. Consider, for example, the set of all even integers. To see that this set has no least element, take an arbitrary even integer $2z$ and notice that $2z - 2 = 2(z-1)$, so $2z - 2$ is also even and is less than z. This argument generalizes to any even integer, so this set has no least element.

[1] You do not have to agree with these definitions, but we must attempt to introduce *some* definition for *"even"* and *"odd"* in order to continue arguing.

[2] How do we know this? What if there is some positive even integer out there such that *no finite number* of 2s added to 0 could reach it? Does the number of 2s being finite even matter?

[3] This is implied by the common meaning of the word *"consecutive"* here.

[4] Why?

[5] For any integer a, we can see that $2a + 2 = 2(a+1)$.

c. "Every non-empty set of natural numbers has a least element." is *true*.

Recall that 0 is the smallest natural number. Consider an arbitrary non-empty set of natural numbers A and let a_0 be one of its elements.[6]
If a_0 is not the least element of A, then there must be an element a_1 such that $a_1 \leqslant a_0 - 1 < a_0$. Now, if a_1 is not the least element of A, there must be an element a_2 such that $a_2 \leqslant a_1 - 1 < a_1$. We can continue asking this question, looking for the least element of A, and after at most a_0 iterations of this process we will exhaust all of the numbers between 0 and a_0. Since there are no natural numbers less than 0, one of the numbers between 0 and a_0 must be the least element of A.

[6] Notice that we know there exists such an element a_0 in the first place because A is non-empty.

d. "Every natural number is boring." is *false*.

For example, 0 has the interesting property $\forall x(0 \leqslant x)$, where our quantification is understood to be over the natural numbers. Since 0 is the only natural number with this property, and since that excites me personally, and since one person's excitement is enough to preclude boringness, 0 is not boring.

e. "No natural number is boring." is *true*.

To show that every natural number is interesting, suppose towards the contrary that there is at least one boring number. Now, consider the set of all boring numbers B. This is a non-empty set of natural numbers, so it has a least element b. This number b is the *smallest* boring number… a very interesting property! This means b is *not* boring, contradicting the fact that b is the smallest *boring* number. Therefore, our assumption that there exists at least one boring number was mistaken, and so there are no boring natural numbers.

2. We will determine the truth value of each of the following sentences.

a. "This sentence is *true*." could be *either true or false but not both*.

I mean… literally just read the sentence. If you believe it, then it's *true.* However, if you don't believe it, then you would say the sentence is *not true,* which would normally imply that the sentence is *false* if we take the classical approach that there are only two truth values and that each is the opposite of the other. This lines up perfectly with your understanding of the sentence, so there wouldn't seem to be any contradiction from your perspective in that case, just as there wouldn't seem to be any contradiction for anyone reading that sentence who *did* believe in it. The sentence cannot hold *both* truth values at the same time, but either choice of truth value can be assigned to the sentence without issue.

b. "This sentence is *false*." is *neither truth nor false*.

Let S be the sentence "S *is* false" and suppose S is *true.* Then, we should believe what S says, and therefore we believe S is *false,* immediately contradicting our assumption. On the other hand, assuming S is *false,* we should then believe the opposite of what S says. This means we believe S is not *false,* again contradicting our assumption. Since we run into contradictions in either case, there is no consistent truth value that can be assigned to that sentence.

c. "The set of all sets that contain themselves contains itself." could be *either true or false but not both.*

We should first suppose that such an object like "the set of all sets that contain themselves" exists and that we can talk about it in a consistent way. Let's give this set the name B. The elements of B are all and only those sets x such that x contains x as an element.

If the given sentence is *true,* then we should believe that U contains itself as an element. By the definition of U, that means that U is an element of U, which agrees with the observation we just made. It seems that no contradictions are introduced.

If we instead suppose that the given sentence is *false,* then we would *not* believe that U contains itself. By the definition of U, we then know U is *not* one of the sets that contains itself, meaning that U does not contain itself. This agrees with our observation that the given sentence is *false,* and again it seems like all is right with the world.

Therefore, much like the sentence in 2.a., this sentence could consistently hold either truth value.

d. "The set of all sets that do not contain themselves does not contain itself." is *neither true nor false.*

Suppose that such a set exists, and call it R. If the sentence is *true,* then R contains R as an element, and therefore R is one of the sets that does not contain itself, contradicting the fact that R contains R.

Alternatively, suppose the sentence is *false.* Then, R is *not* an element of R, so R is not one of the sets that contains themselves. However, since R does not contain itself, it qualifies as one of the elements of R by definition, contradicting our previous observation.

e. "If this sentence is *false,* then 7 is a prime number." is *true.*

Call the sentence above P, so that it says *"If P is* false, *then* 7 *is prime."* If P were *false,* this would imply that 7 is prime. But 7 is actually prime, so the inference that *"if P is* false, *then* 7 *is prime"* is *true,* meaning that P is *true,* contradicting the fact that we assumed P was *false.*

If we instead consider P to be *true,* then P is a conditional statement with a *false* premise, which is *true* by definition, and all is right with the world.

f. "If this sentence is *true,* then 7 is not a prime number." is *neither true nor false.*

Let C be the sentence above. C says *"If C is true, then 7 is not prime."*

First, suppose C is *false,* implying C has a *true* premise and a *false* conclusion because C is an implication. However, the premise of C asserts that C is *true,* contradicting our assumption that C was *false.*
Second, assume C is *true.* This means the premise of C is *true,* and thus we know the conclusion of C must be *true.* This tells us 7 is not prime, which again contradicts the fact that 7 is prime.

Therefore, neither truth value can consistently be assigned to C.

3. We will fix a universe of discourse consisting of all of the characters in Shakespeare's *Macbeth,* and we take for granted the unary predicates κ, ϑ, υ, β, and the binary predicate μ that were previously introduced.

   a. $\kappa(\text{Macbeth}) \land \vartheta(\text{Banquo})$.

   b. $\forall x(\kappa(x) \to \exists y(\mu(y, x)))$.

   c. $\exists x \forall y(\kappa(y) \to (\mu(x, y)))$.

   d. We present two reasonable translations that are logically equivalent.

      $\forall x(\upsilon(x) \to \forall y(\neg\mu(x, y)))$.

      $\neg\exists x(\upsilon(x) \land \exists y(\mu(x, y)))$.

   e. We present two reasonable translations that are logically equivalent.

      $\forall x(\kappa(x) \to \forall y(\vartheta(y) \to \neg\mu(x, y)))$.

      $\neg\exists x \exists y((\kappa(x) \land \vartheta(y)) \land \mu(x, y))$.

   f. We present two reasonable translations that are logically equivalent.

      $\forall x(\beta(x) \to \neg\mu(x, \text{Macbeth}))$.

      $\neg\exists x(\beta(x) \land \mu(x, \text{Macbeth}))$.

   g. $\forall x(\kappa(x) \to \vartheta(x))$.

   h. $\neg\kappa(\text{Banquo}) \land \exists x(\mu(x, \text{Banquo}))$.