Problem Set 7

Discrete Structures

Due on the 17[th] day of April of the year of our Lord 2026 at 11:59 pm

As always, you may rely on any statement we have previously proven in lectures and problem sets. You should solve the problems *in order*; solutions to earlier problems may be applied as theorems in the proofs of later problems, but *not vice versa*. In this problem set, whenever you *define a function*, you are ***not*** required to *prove* that the thing you've defined is in fact a function (as long as you *have* actually defined a function). You may also rely on the *Cantor-Schröder-Bernstein* theorem as stated below.

Theorem: Cantor-Schröder-Bernstein.
$\forall x \forall y \Big(\big(|x| \leqslant |y| \wedge |y| \leqslant |x| \big) \Rightarrow |x| = |y| \Big)$.

We define a set X to be *finite iff* $(\exists n \in \mathbb{N})\big(|X| = |n|\big)$, meaning that X can be put in bijection with the set $\{0, 1, \ldots n-1\}$ for some $n \in \mathbb{N}$. In such a situation, we will define

$|X|$ the *cardinality of X* to be $|X| := n$, so saying $|X| \in \mathbb{N}$ is equivalent to saying "X is finite."

Given this definition, you may also rely on the following theorems in this problem set.

Lemma.
If X is a finite set and Y is a set such that $Y \subseteq X$, then Y is also finite.

Theorem.
For any finite sets X and Y, if $Y \subseteq X$, then $|X \setminus Y| = |X| - |Y|$.

Corollary.
For any nonempty finite set X, and for any $x \in X$, we have $\big|X \setminus \{x\}\big| = |X| - 1$.

Given sets A, B, C and functions $f : A \to B$ and $g : B \to C$, we define the *composition* of f

$g \circ f$ with g to be the function $g \circ f : A \to C$ given by $\big(g \circ f\big)(a) := g\big(f(a)\big)$ for all $a \in A$.

1. a. Show that $\forall X \forall Y \big(X \subseteq Y \Rightarrow |X| \leqslant |Y| \big)$.

   b. Let X and Y be sets such that $X \neq \varnothing$ and $Y \neq \varnothing$, and let $f : X \to Y$ be an injection. Prove that there exists a function $g : Y \to X$ such that $g \circ f = \mathtt{id}_X$.[1]

2. a. Prove that $\forall A \forall B \forall C \Big(\big(|A| \leqslant |B| \wedge |B| \leqslant |C| \big) \Rightarrow |A| \leqslant |C| \Big)$.

   b. Prove that $\forall A \forall B \forall C \Big(\big(|A| \geqslant |B| \wedge |B| \geqslant |C| \big) \Rightarrow |A| \geqslant |C| \Big)$.

   c. Prove that $\forall A \forall B \forall C \Big(\big(|A| = |B| \wedge |B| = |C| \big) \Rightarrow |A| = |C| \Big)$.

3. For any finite sets A and B, prove that $|A \cup B| = |A| + |B|$ when $A \cap B = \varnothing$.

4. For any finite set X, show that $\big|\mathbb{P}(X)\big| = 2^{|X|}$.[2]

[1] This problem is asking you to prove that every injective function has a left inverse. The dual statement—every surjective function has a right inverse—is equivalent to the *axiom of choice*.

[2] *Hint:* try induction.