Solution Set 6

Discrete Structures

1st day of April of the year of our Lord 2026

1. We will prove that $(\forall a,b,x,y,n \in \mathbb{Z})\big((n \mid x \wedge n \mid y) \;\Rightarrow\; (n \mid ax+by)\big)$.

   Proof. Let $a,b,x,y,n \in \mathbb{Z}$ and assume that $n \mid x$ and $n \mid y$. By definition, this means there exist $k_x \in \mathbb{Z}$ such that $n \cdot k_x = x$, and there exists $k_y \in \mathbb{Z}$ such that $n \cdot k_y = y$.

   $$n \cdot k_x = x \;\Rightarrow\; n \cdot (a \cdot k_x) = a \cdot x$$
   $$n \cdot k_y = y \;\Rightarrow\; n \cdot \left(b \cdot k_y\right) = b \cdot y$$

   By adding these equations together and factoring out n, we obtain the following.

   $$a \cdot x + b \cdot y = \big(n \cdot (a \cdot k_x)\big) + \Big(n \cdot \left(b \cdot k_y\right)\Big) = n \cdot \left(a \cdot k_x + b \cdot k_y\right)$$

   Since $a \cdot k_x + b \cdot k_y \in \mathbb{Z}$, we conclude that $n \mid a \cdot x + b \cdot y$ by definition.

   QED

2. a. We will prove $(\forall z \in \mathbb{Z})\big(2 \nmid z \vee 2 \nmid (z-1)\big)$.

   Proof. Let $z \in \mathbb{Z}$ and suppose, towards a contradiction, that $2 \mid z$ and $2 \mid z-1$. Then we know there exist $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$ such that $2k = z$ and $2\ell = z-1$ by definition. From this, we can see that $z = 2\ell + 1$, so that $2k = 2\ell + 1$. Observe.

   $$\begin{aligned} 2k = 2\ell + 1 \;&\Rightarrow\; 2k - 2\ell = 1 \\ &\Rightarrow\; 2\big(k - \ell\big) = 1 \end{aligned}$$

   Since $k - \ell \in \mathbb{Z}$, this implies that $2 \mid 1$, so that $|2| \leqslant |1|$. The fact that $1 \in \mathbb{N}$ and $2 \in \mathbb{N}$ implies $0 \leqslant 1$ and $0 \leqslant 2$, so we have that $1 = |1|$ and $2 = |2|$, so that $2 \leqslant 1$, from which we can deduce $2 = 1 \vee 2 \in 1$ by definition.

   QED

   b. We will prove $(\forall z \in \mathbb{Z})\big((z \text{ is even}) \;\Leftrightarrow\; (z+1 \text{ is odd})\big)$.

   Proof. Let $z \in \mathbb{Z}$ and observe.

z is even	$\Leftrightarrow$	$2 \mid z$	by definition of *"even"*
	$\Leftrightarrow$	$(\exists k \in \mathbb{Z})(2k = z)$	by definition of divisibility
	$\Leftrightarrow$	$(\exists k \in \mathbb{Z})(2k = (z+1) - 1)$	since $(z+1) - 1 = z$
	$\Leftrightarrow$	$2 \mid (z+1) - 1$	by definition of divisibility
	$\Leftrightarrow$	z is odd	by definition of *"odd"*

   Therefore, z is even *iff* z is odd.

   QED

   c. We will prove $(\forall n \in \mathbb{N})\big((n \text{ is even}) \vee (n \text{ is odd})\big)$ by *weak induction.*

   Proof.

   Basis Step:
   Notice that $2 \cdot 0 = 0$, so that $2 \mid 0$ by definition, showing us 0 is even.

Inductive Step:
Let $k \in \mathbb{N}$ and assume that $\big(k \text{ is even}\big) \vee \big(k \text{ is odd}\big)$. We take two cases.[1]

[1] *inductive hypothesis*

Case 1:
Suppose k is even, meaning $2 \mid k$ by definition. Since $\big(k+1\big) - 1 = k$, this clearly implies that $2 \mid \big(k+1\big) - 1$, so that $k+1$ is odd by definition. Thus, $\big(k+1 \text{ is even}\big) \vee \big(k+1 \text{ is odd}\big)$.

Case 2:
Suppose that k is odd, meaning $2 \mid k-1$ by definition. This means there exists $\ell \in \mathbb{Z}$ such that $2\ell = k-1$ by definition. Adding 2 to both sides yields $2\ell + 2 = \big(k-1\big) + 2$, which simplifies to $2(\ell+1) = k+1$. Since $\ell + 1 \in \mathbb{Z}$, we can deduce $2 \mid k+1$, telling us $k+1$ is even by definition. Thus, $\big(k+1 \text{ is even}\big) \vee \big(k+1 \text{ is odd}\big)$.

In either case, we have shown that either $k+1$ is even or $k+1$ is odd.

Therefore, we can conclude that $(\forall n \in \mathbb{N})\Big(\big(n \text{ is even}\big) \vee \big(n \text{ is odd}\big)\Big)$.

QED

3. We will show $(\exists m \in \mathbb{N})(\forall n \in \mathbb{N})\big(n \geqslant m \;\Rightarrow\; n^2 < 2^n\big)$.

Proof. Letting $m := 5$, we prove $(\forall n \in \mathbb{N})\big(n \geqslant 5 \;\Rightarrow\; n^2 < 2^n\big)$ by *weak induction*.[2]

[2] Since we are only proving $n^2 < 2^n$ when $n \geqslant 5$, we know that the statement we are trying to prove is *vacuously true (because it is a conditional with a false premise)* whenever $n \in \{0, 1, 2, 3, 4\}$. It should therefore be reasonable to *"start"* our induction by taking 5 as our basis step because that's the *"real"* basis that we would have to explicitly prove anyways in the inductive step.

Basis Step:
Observe that $5^2 = 25 < 32 = 2^5$.

Inductive Step:
Let $k \in \mathbb{N}$, and suppose $k \geqslant 5 \;\Rightarrow\; k^2 < 2^k$.[3] Assume $k+1 \geqslant 5$, so that $k+1 = 5$ or $k+1 > 5$. We now take two cases.

[3] *inductive hypothesis*

Case 1:
If $k+1 = 5$, we immediately have $\big(k+1\big)^2 < 2^{k+1}$ thanks to our basis step.

Case 2:
Suppose $k+1 > 5$. Notice that $k+1 > 5 \;\Leftrightarrow\; k > 4 \;\Leftrightarrow\; k \geqslant 5 \;\Leftrightarrow\; k-2 \geqslant 3$ because $k \in \mathbb{N}$. So, we know $k \geqslant 5$ and $k-2 \geqslant 3$. The fact that multiplication is monotonic then tells us that $k\big(k-2\big) \geqslant 5 \cdot 3 = 15 > 1$. Now, observe.

$$k\big(k-2\big) > 1 \;\Rightarrow\; k^2 - 2k > 1 \;\Rightarrow\; k^2 > 2k+1$$

Recalling that $k \geqslant 5$, we apply our *inductive hypothesis* to see $k^2 < 2^k$. Taking this in concert with $k^2 > 2k+1$, we can now make the following deduction.

$$\begin{aligned}
\big(k+1\big)^2 &= k^2 + 2k + 1 && \text{by distributing several times and } 1+1=2 \\
&< k^2 + k^2 && \text{since } k^2 < 2k+1 \\
&< 2^k + 2^k && \text{by the \textit{inductive hypothesis}} \\
&< 2 \cdot 2^k && \text{by distributivity, commutativity, and } 1+1=2 \\
&< 2^{k+1} && \text{by definition of exponentiation}
\end{aligned}$$

Thus, in either case, we have that $\big(k+1\big)^2 < 2^{k+1}$.

Therefore, we conclude that $(\forall n \in \mathbb{N})\big(n \geqslant 5 \;\Rightarrow\; n^2 < 2^n\big)$ as desired.

QED

4. We will show $(\forall n \in \mathbb{N})(n \geqslant 4 \Rightarrow 2^n < n!)$ by *weak induction.*

Proof. As with the previous problem, we will perform induction on $\mathbb{N} \setminus \{0,1,2,3\}$.

Basis Step:
Observe the following sequences of computations.

$$\begin{aligned} 2^4 &= 2 \cdot 2^3 \\ &= 2 \cdot \left(2 \cdot 2^2\right) \\ &= 2 \cdot \left(2 \cdot \left(2 \cdot 2^1\right)\right) \\ &= 2 \cdot \left(2 \cdot \left(2 \cdot \left(2 \cdot 2^0\right)\right)\right) \\ &= 2 \cdot \left(2 \cdot \left(2 \cdot (2 \cdot 1)\right)\right) \\ &= 2 \cdot \left(2 \cdot (2 \cdot 2)\right) \\ &= 2 \cdot (2 \cdot 4) \\ &= 2 \cdot 8 \\ &= 16 \end{aligned} \qquad \begin{aligned} 4! &= 4 \cdot (3!) \\ &= 4 \cdot \left(3 \cdot (2!)\right) \\ &= 4 \cdot \left(3 \cdot \left(2 \cdot (1!)\right)\right) \\ &= 4 \cdot \left(3 \cdot \left(2 \cdot (1 \cdot 0!)\right)\right) \\ &= 4 \cdot \left(3 \cdot \left(2 \cdot (1 \cdot 1)\right)\right) \\ &= 4 \cdot \left(3 \cdot (2 \cdot 1)\right) \\ &= 4 \cdot (3 \cdot 2) \\ &= 4 \cdot 6 \\ &= 24 \end{aligned}$$

Since $16 < 24$, we conclude that $2^4 < 4!$ as required.

Inductive Step:
Let $k \in \mathbb{N} \setminus \{0,1,2,3\}$, so that $k \geqslant 4$, and assume $2^k < k!$.[4] First, we can notice that $k+1 > k \geqslant 4 > 2$; this then implies $2 \cdot k! \leqslant \left(k+1\right) \cdot k!$ by the monotonicity of multiplication. We can then observe the following sequence of inequalities.

[4] *inductive hypothesis*

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k && \text{by definition of exponentiation} \\ &< 2 \cdot k! && \text{by the } \textit{inductive hypothesis} \\ &\leqslant \left(k+1\right) \cdot k! && \text{since } 2 \cdot k! \leqslant \left(k+1\right) \cdot k! \\ &= \left(k+1\right)! && \text{by definition of the factorial function} \end{aligned}$$

Thus, we arrive at $2^{k+1} < \left(k+1\right)!$, concluding our inductive step.

Therefore, $(\forall n \in \mathbb{N})(n \geqslant 4 \Rightarrow 2^n < n!)$.

QED

5. We will show $(\forall n \in \mathbb{N})\left(\displaystyle\sum_{i=1}^{n} \mathcal{F}(2i-1) = \mathcal{F}(2n)\right)$ by *weak induction.*

Proof.

Basis Step:
First, notice $\sum_{i=1}^{0} \mathcal{F}(2i-1) = 0$ by definition since $0 < 1$. Second, we can see $\mathcal{F}(2 \cdot 0) = \mathcal{F}(0) = 0$ by definition. Therefore, $\sum_{i=1}^{0} \mathcal{F}(2i-1) = 0 = \mathcal{F}(2 \cdot 0)$.

Inductive Step:
Let $k \in \mathbb{N}$ and assume $\sum_{i=1}^{k} \mathcal{F}(2i-1) = \mathcal{F}\left(2k\right)$.[5] We can now simply observe.

[5] *inductive hypothesis*

$$
\begin{aligned}
\sum_{i=1}^{k+1} \mathcal{F}(2i-1) &= \left(\sum_{i=1}^{k} \mathcal{F}(2i-1)\right) + \mathcal{F}\big(2(k+1)-1\big) && \text{by definition of } \textstyle\sum \\
&= \mathcal{F}(2k) + \mathcal{F}\big(2(k+1)-1\big) && \text{by the \textit{inductive hypothesis}} \\
&= \mathcal{F}(2k) + \mathcal{F}(2k+1) && \text{because } 2(k+1)-1 = 2k+1 \\
&= \mathcal{F}(2k+2) && \text{by definition of } \mathcal{F}(\square) \\
&= \mathcal{F}\big(2(k+1)\big) && \text{since } 2k+2 = 2(k+1)
\end{aligned}
$$

Thus, we have that $\sum_{i=1}^{k+1} \mathcal{F}(2i-1) = \mathcal{F}\big(2(k+1)\big)$ as desired.

We therefore conclude $\sum_{i=1}^{n} \mathcal{F}(2i-1) = \mathcal{F}(2n)$ for all $n \in \mathbb{N}$.

QED