

Problem Set 6

Discrete Structures

Due on the 1st day of April of the year of our Lord 2026 at 11:59 pm

As always, you may rely on any statement we have previously proven in lectures and problem sets. You should solve the problems *in order*; solutions to earlier problems may be applied as theorems in the proofs of later problems, but *not vice versa*.

Explicitly: you may assume all of the theorems discussed in lecture up until (and including) the 23rd of March, as well as all content up until (and including) *clavicula 08*.

1. Prove $(\forall a, b, x, y, n \in \mathbb{Z})((n \mid x \wedge n \mid y) \Rightarrow (n \mid ax + by))$.
2. We say an integer $z \in \mathbb{Z}$ is *even* $:\Leftrightarrow 2 \mid z$, and we say z is *odd* $:\Leftrightarrow 2 \mid z - 1$.
 - a. Prove that no integer can be both even and odd at the same time.
 - b. Prove that $(\forall z \in \mathbb{Z})((z \text{ is even}) \Leftrightarrow (z + 1 \text{ is odd}))$.
 - c. Prove that $(n \text{ is even}) \vee (n \text{ is odd})$ for all $n \in \mathbb{N}$.¹
3. Show that $(\exists m \in \mathbb{N})(\forall n \in \mathbb{N})(n \geq m \Rightarrow n^2 < 2^n)$.
4. The *factorial* function $! : \mathbb{N} \rightarrow \mathbb{N}$ is the recursive function given below for all $n \in \mathbb{N}$.

$$\begin{aligned} 0! &:= 1 \\ (n+1)! &:= (n!) \cdot (n+1) \end{aligned}$$

Prove that $(\forall n \in \mathbb{N})(n \geq 4 \Rightarrow 2^n < n!)$.

5. The *Fibonacci sequence* is the function $\mathcal{F} : \mathbb{N} \rightarrow \mathbb{N}$ defined below for all $n \in \mathbb{N}$.

$$\begin{aligned} \mathcal{F}(0) &:= 0 \\ \mathcal{F}(1) &:= 1 \\ \mathcal{F}(n+2) &:= \mathcal{F}(n+1) + \mathcal{F}(n) \end{aligned}$$

Prove that $\sum_{i=1}^n \mathcal{F}(2i-1) = \mathcal{F}(2n)$ for all $n \in \mathbb{N}$.

¹Once you have proven this result, we can easily see that z is even $\Leftrightarrow -z$ is even for all $z \in \mathbb{Z}$, and thus we will know every integer is either even or odd but not both.