Solution Set 4

Discrete Structures

24th day of February of the year of our Lord 2026

1. a. We will prove $\forall x \forall y \big(\mathbb{P}(x) \cup \mathbb{P}(y) \subseteq \mathbb{P}(x \cup y)\big)$.

Proof. Let x and y be sets. Let t be a set and assume $t \in \mathbb{P}(x) \cup \mathbb{P}(y)$. By definition, this means $t \in \mathbb{P}(x)$ or $t \in \mathbb{P}(y)$, which implies $t \subseteq x$ or $t \subseteq y$ by definition.

Case 1:
Suppose $t \subseteq x$. We then know $t \subseteq x \cup y$,[1] so that $t \in \mathbb{P}(x \cup y)$ by definition.

Case 2:
Suppose $t \subseteq y$, so that $t \subseteq y \cup x$.[2] Since $\forall z (z \in x \lor z \in y \iff z \in y \lor z \in x)$, we know $y \cup x = x \cup y$ by the *axiom of extensionality*. This implies $t \subseteq x \cup y$, so that $t \in \mathbb{P}(x \cup y)$ by definition.

QED

b. We will prove $\exists x \exists y \big(\mathbb{P}(x) \cup \mathbb{P}(y) \neq \mathbb{P}(x \cup y)\big)$.

Proof. Recall that we proved $\varnothing$ exists. By the *axiom of pairing,* we know $\{\varnothing, \varnothing\}$ exists, and $\{\varnothing, \varnothing\} = \{\varnothing\}$ by the *axiom of extensionality.* Similarly, the *axiom of pairing* tells us $\big\{\{\varnothing\}, \{\varnothing\}\big\}$ exists, and $\big\{\{\varnothing\}, \{\varnothing\}\big\} = \big\{\{\varnothing\}\big\}$ follows from the *axiom of extensionality.* Let $x := \{\varnothing\}$ and $y := \big\{\{\varnothing\}\big\}$. For any set t, we can see the following:

$$\begin{aligned} t \in \big\{\varnothing, \{\varnothing\}\big\} &\iff t = \varnothing \lor t = \{\varnothing\} && \text{by definition} \\ &\iff t \in \{\varnothing\} \lor t \in \big\{\{\varnothing\}\big\} && \text{by definition} \\ &\iff t \in x \cup y && \text{by definition} \end{aligned}$$

Since we know $x \cup y \subseteq x \cup y$,[3] this implies $\big\{\varnothing, \{\varnothing\}\big\} \in \mathbb{P}(x \cup y)$ by definition. However, notice that $\varnothing \notin y$ because $\varnothing \neq \{\varnothing\}$ by the *axiom of extensionality.* This implies $\big\{\varnothing, \{\varnothing\}\big\} \not\subseteq y$.

$$\begin{aligned} \varnothing \neq \{\varnothing\} &\Rightarrow \{\varnothing\} \notin \{\varnothing\} \land \varnothing \notin \big\{\{\varnothing\}\big\} && \text{by def. of } \textit{set roster notation} \\ &\Rightarrow \{\varnothing\} \notin x \land \varnothing \notin y && \text{by def. of } x \text{ and } y \\ &\Rightarrow \big\{\varnothing, \{\varnothing\}\big\} \not\subseteq x \land \big\{\varnothing, \{\varnothing\}\big\} \not\subseteq y && \text{by def. of } \subseteq \\ &\Rightarrow \big\{\varnothing, \{\varnothing\}\big\} \notin \mathbb{P}(x) \cup \mathbb{P}(y) && \text{by def. of } \mathbb{P}(\cdot) \end{aligned}$$

Therefore, we conclude $\mathbb{P}(x) \cup \mathbb{P}(y) \neq \mathbb{P}(x \cup y)$ by the *axiom of extensionality.*

QED

c. We will prove $\forall x \forall y \big(x \cap y = y \Rightarrow y \in \mathbb{P}(x)\big)$.

Proof. Let x and y be sets, and assume $x \cap y = y$. Let t be a set such that $t \in y$. Then, by the *axiom of extensionality,* we know $t \in x \cap y$, implying $t \in x \land t \in y$ by definition. From this, we can see $t \in x$. Thus, $y \subseteq x$ by definition, from which we immediately see $y \in \mathbb{P}(x)$ follows by definition.

QED

[1] *Problem 1.c.* on *problem set 3.*

[2] *Problem 1.c.* on *problem set 3.*

[3] *Lemma 3.1* from *clavicula 3.*

d. We will prove $\forall x \forall y \big(y \in \mathbb{P}(x) \Rightarrow x \cap y = y\big)$.

Proof. Let x and y be sets, and assume $y \in \mathbb{P}(x)$. This means $y \subseteq x$ by definition. We will now show that $y \subseteq x \cap y$. Let t be a set, and assume $t \in y$. Then, since $y \subseteq x$, we know $t \in x$ by definition. This implies $t \in x \wedge t \in y$, so that $t \in x \cap y$ by definition. Therefore, $y \subseteq x \cap y$ by definition. Since we also know $x \cap y \subseteq y$,[4] we can conclude $x \cap y = y$.[5]

[4] *Problem 1.b.* on *problem set 3*.

[5] *Theorem 2.1* from *clavicula 2*.

QED

2. a. We will prove that $\forall x\Big(\big((x \cup \{x\}) \setminus \{x\}\big) = x\Big)$.

Proof. Let x be an arbitrary set. Let t be an arbitrary set.

Fragment 1: Forwards.
Suppose $t \in \big(x \cup \{x\}\big) \setminus \{x\}$, so that $t \in x \cup \{x\}$ and $t \notin \{x\}$ by definition. From our first assumption, we can see $t \in x \vee t \in \{x\}$ by definition. Using our second assumption, we derive $t \in x$.[6]

[6] This is an application of the *disjunctive syllogism*.

Fragment 2: Backwards.
Suppose $t \in x$, which implies $t \in x \vee t \in \{x\}$, so $t \in x \cup \{x\}$ by definition. Recalling that $\forall w(w \notin w)$,[7] we can see $t \neq x$ from our assumption $t \in x$. Therefore, $t \in \big(x \cup \{x\}\big) \setminus \{x\}$ by definition.

[7] *Theorem 5.1* from *clavicula 5*.

Therefore, $\big(x \cup \{x\}\big) \setminus \{x\} = x$ by the *axiom of extensionality.*

QED

b. We will prove that $x \triangle y = \big(x \cup y\big) \setminus \big(x \cap y\big)$ for all sets x and y.

Proof. Let x and y be sets. Let t be an arbitrary set and observe the following.

$$
\begin{aligned}
t \in x \triangle y \quad &\Leftrightarrow \quad t \in \big(x \setminus y\big) \cup \big(y \setminus x\big) && \text{by definition}\\
&\Leftrightarrow \quad t \in \Big\{z \,\Big|\, \big(z \in x \setminus y\big) \vee \big(z \in y \setminus x\big)\Big\} && \text{by definition}\\
&\Leftrightarrow \quad t \in \Big\{z \,\Big|\, \big(z \in x \wedge z \notin y\big) \vee \big(z \in y \wedge z \notin x\big)\Big\} && \text{by definition}\\
&\Leftrightarrow \quad t \in \Big\{z \,\Big|\, \big(z \in x \vee z \in y\big) \wedge \big(z \notin x \vee z \notin y\big)\Big\} &&\\
&\Leftrightarrow \quad t \in \Big\{z \,\Big|\, \big(z \in x \vee z \in y\big) \wedge \neg\big(z \in x \wedge z \in y\big)\Big\} &&\\
&\Leftrightarrow \quad t \in \Big\{z \,\Big|\, \big(z \in x \cup y\big) \wedge \big(z \notin x \cap y\big)\Big\} && \text{by definition}\\
&\Leftrightarrow \quad t \in \big(x \cup y\big) \setminus \big(x \cap y\big) && \text{by definition}
\end{aligned}
$$

Therefore, $x \triangle y = \big(x \cup y\big) \setminus \big(x \cap y\big)$ by the *axiom of extensionality.*

QED

3. a. We will show that $\forall x\big(x \neq x \cup \{x\}\big)$.

Proof. Let x be a set. Towards a contradiction, suppose $x = x \cup \{x\}$. Recall that $x \in \{x\}$, so $x \in x \cup \{x\}$ by definition. This implies that $x \in x$ by the *axiom of extensionality.* However, we know $\forall w(w \notin w)$.[8] $\lightning$ Therefore, $x \neq x \cup \{x\}$.

[8] *Theorem 5.1* from *clavicula 5*.

QED

b. We will show that $\forall x \forall y\big(x \neq y \Rightarrow x \cup \{x\} \neq y \cup \{y\}\big)$.

Proof. Let x and y be sets such that $x \neq y$. Towards a contradiction, suppose that $x \cup \{x\} = y \cup \{y\}$. Notice that $x \in \{x\}$ by definition and observe.

$$\begin{aligned}
x \in \{x\} &\Rightarrow x \in x \vee x \in \{x\} && \text{by } \textit{disjunction introduction} \\
&\Rightarrow x \in \big\{z \,\big|\, z \in x \vee z \in \{x\}\big\} && \text{by definition} \\
&\Rightarrow x \in x \cup \{x\} && \text{by definition} \\
&\Rightarrow x \in y \cup \{y\} && \text{by the } \textit{axiom of extensionality} \\
&\Rightarrow x \in \Big\{z \,\Big|\, z \in y \vee z \in \{y\}\Big\} && \text{by definition} \\
&\Rightarrow x \in y \vee x \in \{y\} && \text{by definition} \\
&\Rightarrow x \in y \vee x = y && \text{by definition} \\
&\Rightarrow x \in y && \text{because } x \neq y
\end{aligned}$$

This sequence of implications forces us to infer that $x \in y$. If we also notice that $y \in \{y\}$ by definition, we are compelled to draw a similar conclusion about y.

$$\begin{aligned}
y \in \{y\} &\Rightarrow y \in y \vee y \in \{y\} && \text{by } \textit{disjunction introduction} \\
&\Rightarrow y \in \Big\{z \,\Big|\, z \in y \vee z \in \{y\}\Big\} && \text{by definition} \\
&\Rightarrow y \in y \cup \{y\} && \text{by definition} \\
&\Rightarrow y \in x \cup \{x\} && \text{by the } \textit{axiom of extensionality} \\
&\Rightarrow y \in \big\{z \,\big|\, z \in x \vee z \in \{x\}\big\} && \text{by definition} \\
&\Rightarrow y \in x \vee x \in \{x\} && \text{by definition} \\
&\Rightarrow y \in x \vee y = x && \text{by definition} \\
&\Rightarrow y \in x && \text{because } x \neq y
\end{aligned}$$

So now we have both $x \in y$ and $y \in x$. However, we also happen to know $\forall a \forall b\big(a \in b \ \Rightarrow\ b \notin a\big)$,[9] which we proved using the *axiom of regularity*. ↯

[9] *Theorem 5.2* from *clavicula 5*.

Therefore, $x \cup \{x\} \neq y \cup \{y\}$.

QED

4. We will show that the Cartesian product of any two sets exists. To be clear, we will prove the formal sentence $\forall x \forall y \exists z\big(z = x \times y\big)$.

Proof. Let X and Y be sets. Recall that $X \cup Y$ exists.[10] We can then apply the *axiom of power* to see $\mathbb{P}(X \cup Y)$ exists; applying this axiom again shows us $\mathbb{P}\big(\mathbb{P}(X \cup Y)\big)$ exists. Now, by the *axiom schema of separation,* the set $\mathfrak{P}$ shown below exists.

[10] *Theorem 4.1* from *clavicula 4*.

$$\mathfrak{P} = \Big\{z \,\Big|\, z \in \mathbb{P}\big(\mathbb{P}(X \cup Y)\big) \wedge \exists x \exists y\big(x \in X \wedge y \in Y \wedge z = (x,y)\big)\Big\}$$

We will prove $\mathfrak{P} = X \times Y$ by setting up an invocation of *extensionality.* Let t be a set.

Fragment 1: Forwards.
Suppose $t \in \mathfrak{P}$. This means $t \in \mathbb{P}\big(\mathbb{P}(X \cup Y)\big)$ and that there exist sets x and y such that $x \in X$ and $y \in Y$ and $t = (x,y)$. However, this means precisely that $t \in \Big\{(a,b) \,\Big|\, a \in X \wedge b \in Y\Big\}$, so $t \in X \times Y$ by definition.

Fragment 2: Backwards.
Suppose $t \in X \times Y$, so that $t = (x,y)$ for some $x \in X$ and $y \in Y$. Recall, by definition, $t = (x,y) = \Big\{\{x\}, \{x,y\}\Big\}$. We have the following inference by definition.

$$x \in X \ \Rightarrow\ \{x\} \subseteq X \ \Rightarrow\ \{x\} \in \mathbb{P}(X)$$

Below, we derive an analogous inference to the one above; please observe.

$$\begin{aligned} x \in X \wedge y \in Y &\Rightarrow (x \in X \cup Y) \wedge \big(y \in X \cup y\big) \\ &\Rightarrow \{x, y\} \subseteq X \cup Y && \text{by definition} \\ &\Rightarrow \{x, y\} \in \mathbb{P}(X \cup Y) && \text{by definition} \end{aligned}$$

Therefore, $\big\{\{x\}, \{x, y\}\big\} \subseteq \mathbb{P}(X \cup Y)$ by definition. This leads us to the important realization that $\big\{\{x\}, \{x, y\}\big\} \in \mathbb{P}\big(\mathbb{P}(X \cup Y)\big)$ by definition, which implies that $t \in \mathbb{P}\big(\mathbb{P}(X \cup Y)\big)$. Since we knew that $\exists a \exists b \big(a \in X \wedge b \in Y \wedge t = (a, b)\big)$ already, we can now conclude $t \in \mathfrak{P}$ by definition.

Therefore, by the *axiom of extensionality*, we utter $\mathfrak{P} = X \times Y$ and rest.

QED