Problem Set 4

Discrete Structures

Due on the 24th day of February of the year of our Lord 2026 at 11:59 pm

You may rely on Axioms 1, 2, 3, 4, 5, and 6[1] for this problem set. Since we do not *formally* understand Axiom 0—the *axiom of infinity*—yet, we should not rely on the set $\mathbb{N}$ or the concept of a *natural number*. The first set whose existence we formally recognize is $\varnothing$.

[1]...and any theorems we have proven from these axioms so far...

As always, you may rely on any statement we have previously proven in lectures and problem sets. You should solve the problems *in order;* solutions to earlier problems may be applied as theorems in the proofs of later problems, but *not vice versa.*

1. For any set x, the *power set of a set x* is given by $\mathbb{P}(x) := \{y \mid y \subseteq x\}$.
   a. Prove $\forall x \forall y\big(\mathbb{P}(x) \cup \mathbb{P}(y) \subseteq \mathbb{P}(x \cup y)\big)$.
   b. Prove $\exists x \exists y\big(\mathbb{P}(x) \cup \mathbb{P}(y) \neq \mathbb{P}(x \cup y)\big)$.
   c. Prove $\forall x \forall y(x \cap y = y \;\Rightarrow\; y \in \mathbb{P}(x))$.
   d. Prove $\forall x \forall y(y \in \mathbb{P}(x) \;\Rightarrow\; x \cap y = y)$.
2. For any sets x and y, the *relative complement of y in x* is $x \setminus y := \{z \mid z \in x \wedge z \notin y\}$.[2] We also define the *symmetric difference of x and y* by $x \triangle y := (x \setminus y) \cup (y \setminus x)$.
   a. Prove that $\forall x\Big(\big((x \cup \{x\}) \setminus \{x\}\big) = x\Big)$.
   b. Prove that $x \triangle y = (x \cup y) \setminus (x \cap y)$ for all sets x and y.
3. Recall that the *axiom of regularity* asserts $\forall x\big(x \neq \varnothing \;\Rightarrow\; \exists y(y \in x \wedge x \cap y = \varnothing)\big)$.
   a. Show that $\forall x(x \neq x \cup \{x\})$.
   b. Show that $\forall x \forall y\big(x \neq y \;\Rightarrow\; x \cup \{x\} \neq y \cup \{y\}\big)$.
4. We define *Cartesian product of two sets X and Y* by $X \times Y := \big\{(x, y) \;\big|\; x \in X \wedge y \in Y\big\}$. Show that the Cartesian product of any two sets exists.

[2]This is also called the *difference* between x and y, and it is sometimes read as either *"x minus y"* or *"x without y."*