Solution Set 3

Discrete Structures

15^{th} day of February of the year of our Lord 2026

1. a. We will prove $\forall x \forall y \forall z\big((x \subseteq y \land y \subseteq z) \Rightarrow x \subseteq z\big)$.

   Proof. Let x, y, and z be sets, and assume that $x \subseteq y$ and $y \subseteq z$.
   Now, let t be an arbitrary set, and assume $t \in x$. Since $x \subseteq y$, we know that $\forall w(w \in x \Rightarrow w \in y)$ by definition, so that $t \in y$. Since $y \subseteq z$, we similarly have $\forall w(w \in y \Rightarrow w \in z)$ by definition, so that $t \in z$.

   Therefore, $\forall w(w \in x \Rightarrow w \in z)$, so that $x \subseteq z$ by definition.

   QED

   b. We will prove $\forall x \forall y(x \cap y \subseteq x)$.

   Proof. Let x and y be sets. Let t be an arbitrary set, and assume $t \in x \cap y$. By definition,[1] this means $t \in \{z \mid z \in x \land z \in y\}$, from which $t \in x \land t \in y$ follows by definition.[2] This immediately implies $t \in x$.

   [1] $x \cap y := \{z \mid z \in x \land z \in y\}$

   [2] *set comprehension notation*

   Therefore, $\forall w(w \in x \cap y \Rightarrow w \in x)$, showing that $x \cap y \subseteq x$ by definition.

   QED

   c. We will prove $\forall x \forall y(x \subseteq x \cup y)$.

   Proof. Let x and y be sets. Let t be an arbitrary set, and assume $t \in x$. Then, clearly, we can see $t \in x \lor t \in y$. This implies $t \in \{z \mid z \in x \lor z \in y\}$ by definition,[3] from which follows $t \in x \cup y$ by definition.[4]

   [3] *set comprehension notation*

   Therefore, $\forall w(w \in x \Rightarrow w \in x \cup y)$, letting us conclude $x \subseteq x \cup y$ by definition.

   [4] $x \cup y := \{z \mid z \in x \lor z \in y\}$

   QED

   d. We will prove $\forall x \forall y(x \cap y \subseteq x \cup y)$.

   Proof. Let x and y be sets. Recall that $x \cap y \subseteq x$[5] and $x \subseteq x \cup y$.[6] Then, since $\forall a \forall b \forall c\big((a \subseteq b \land b \subseteq c) \Rightarrow a \subseteq c\big)$,[7] we can conclude that $x \cap y \subseteq x \cup y$.

   [5] … by *problem 1.b.* …

   [6] … by *problem 1.c.* …

   [7] … by *problem 1.a.* …

   QED

2. a. We provide two proofs of $\forall x \forall y \exists p\big(p = (x, y)\big)$ with slightly different styles.

   Proof 1. Let x and y be arbitrary sets. By the *axiom of pairing*, we know there exists some set l such that $l = \{x, x\}$. For any z, we can make the following observation:

$$\begin{aligned} z \in \{x, x\} &\Leftrightarrow z = x \lor z = x && \text{by definition of } z \in \{x, x\} \\ &\Leftrightarrow z = x && \text{by } \textit{idempotency} \\ &\Leftrightarrow z \in \{x\} && \text{by definition of } z \in \{x\} \end{aligned}$$

   This shows us that $l = \{x\}$ by the *axiom of extensionality*. The *axiom of pairing* also says there exists some set r such that $r = \{x, y\}$. A third use of the *axiom of pairing* reveals the existence of a set p such that $p = \{l, r\}$. We then see $p = \big\{\{x\}, \{x, y\}\big\}$ by the *axiom of extensionality*, allowing us to conclude $p = (x, y)$ by definition.[8]

   [8] $(x, y) := \big\{\{x\}, \{x, y\}\big\}$.

   QED

Proof 2. Let x and y be arbitrary sets. We know that $\{x,x\}$ and $\{x,y\}$ both exist by the *pairing axiom.* The *axiom of extensionality* makes it clear that $\{x\} = \{x,x\}$. We can again invoke the *axiom of pairing* to see that $\big\{\{x\},\{x,y\}\big\}$ exists. We conclude by observing that $\big\{\{x\},\{x,y\}\big\} = (x,y)$ by definition.

QED

b. We will prove $\forall a \forall b \forall x \forall y \big((a,b) = (x,y) \iff (a = x \land b = y)\big)$.

Proof. Let a, b, x, and y be sets. To show $(a,b) = (x,y) \iff (a = x \land b = y)$, we must prove both the forward $(a,b) = (x,y) \Rightarrow (a = x \land b = y)$ direction and the backward $(a = x \land b = y) \Rightarrow (a,b) = (x,y)$ direction.

Fragment 1: The Forward Direction.
Suppose that $(a,b) = (x,y)$. Then, $\big\{\{a\},\{a,b\}\big\} = \big\{\{x\},\{x,y\}\big\}$ by definition. Since $\{a\} \in \big\{\{a\},\{a,b\}\big\}$, the *axiom of extensionality* implies $\{a\} \in \big\{\{x\},\{x,y\}\big\}$, so $\{a\} = \{x\}$ or $\{a\} = \{x,y\}$ by definition. To show $a = x$, we take two cases.

Case 1:
Suppose that $\{a\} = \{x\}$. We know $a \in \{a\}$ by definition, so $a \in \{x\}$ by the *axiom of extensionality,* and thus $a = x$ by definition.

Case 2:
Suppose that $\{a\} = \{x,y\}$. Since $x \in \{x,y\}$ by definition, we can then see $x \in \{a\}$ by the *axiom of extensionality,* so that $x = a$ by definition.

Therefore, we know $a = x$.[9] Now, since $\{a,b\} \in \big\{\{a\},\{a,b\}\big\}$, the *axiom of extensionality* tells us that $\{a,b\} \in \big\{\{x\},\{x,y\}\big\}$. By definition, this then means that $\{a,b\} = \{x\}$ or $\{a,b\} = \{x,y\}$. To show $b = y$, we again take two cases.

Case 1:
Suppose that $\{a,b\} = \{x\}$. Then, since $b \in \{a,b\}$ by definition, we know $b \in \{x\}$ by the *axiom of extensionality,* which implies $b = x$ by definition. Since $a = x$, this implies $a = b$, from which we can derive $\{a,b\} = \{a\}$ by the *axiom of extensionality,* immediately implying $\big\{\{a\},\{a,b\}\big\} = \big\{\{a\}\big\}$ by the *axiom of extensionality.* If we recall that $\{x,y\} \in \big\{\{x\},\{x,y\}\big\}$ and $\big\{\{a\},\{a,b\}\big\} = \big\{\{x\},\{x,y\}\big\}$, then $\{x,y\} = \{a\}$ by definition.[10] Then, since $y \in \{x,y\}$, we know $y \in \{a\}$ by the *axiom of extensionality,* so that $y = a$. Therefore, $y = b$.[11]

Case 2:
Suppose that $\{a,b\} = \{x,y\}$. Since $y \in \{x,y\}$, we can see $y \in \{a,b\}$ by the *axiom of extensionality,* from which we obtain $y = a$ or $y = b$. In the first case, if $y = a$, then our result $a = x$ from earlier implies $y = x$, so that $\{a,b\} = \{x,y\} = \{y\}$ by the *axiom of extensionality.* Since $b \in \{a,b\}$, we then see $b \in \{y\}$ by the *axiom of extensionality,* so $b = y$ by definition. In the second case, when $y = b$, then we immediately have our desired result $b = y$. Therefore, $y = b$.[12]

Therefore, we know $b = y$.[13] Having proven both $a = x$ and $b = y$, we are finally done with this half of the proof.

[9] This is how we apply *disjunction elimination,* which is also known as *proof by cases.* We reproduce the theorem below as a reminder below. $\varphi \lor \psi,\ \varphi \to \xi,\ \psi \to \xi \vdash \xi$

[10] Because we know that both $\{x,y\} \in \big\{\{a\},\{a,b\}\big\}$ and $\big\{\{a\},\{a,b\}\big\} = \big\{\{a\}\big\}$, we can say $\{x,y\} \in \big\{\{a\}\big\}$, from which $\{x,y\} = \{a\}$ follows by definition.

[11] Recall we proved $a = b$ earlier this case, so this follows by *transitivity of equality.*

[12] This is, yet again, another application of *disjunction elimination,* also known as *proof by cases.*

[13] Guess what? *Disjunction elimination,* also known as *proof by cases.*

Fragment 2: The Backward Direction.
Suppose that $a = x$ and $b = y$. The *axiom of extensionality* and the definition of *set roster notation* then show us the following.

$$\text{For any set } t, \quad t \in \{a\} \;\Leftrightarrow\; t = a \;\Leftrightarrow\; t = x \;\Leftrightarrow\; t \in \{x\}.$$

Thus, $\{a\} = \{x\}$ by the *axiom of extensionality.* Similarly, we can observe.

$$\begin{aligned} \text{For any set } t, \quad t \in \{a, b\} &\Leftrightarrow t = a \lor t = b && \text{by definition} \\ &\Leftrightarrow t = x \lor t = y && \text{because } a = x \text{ and } b = y \\ &\Leftrightarrow t \in \{x, y\} && \text{by definition} \end{aligned}$$

transitivity of equality

Thus, $\{a, b\} = \{x, y\}$ by the *axiom of extensionality.* This of course implies $\big\{\{a\}, \{a, b\}\big\} = \big\{\{x\}, \{x, y\}\big\}$ by the *axiom of extensionality.* We then conclude $(a, b) = \big\{\{a\}, \{a, b\}\big\} = \big\{\{x\}, \{x, y\}\big\} = (x, y)$ by definition.

Therefore, having proven both the forward and backward directions, we can finally conclude that $(a, b) = (x, y) \;\Leftrightarrow\; (a = x \land b = y)$ as desired.

QED

3. a. We will show that $\forall x\big(x \cup \{x\} \neq \varnothing\big)$.

Proof. Let x be an arbitrary set. Observe that $x \in \{x\}$ by definition.[14] This implies that $x \in x \lor x \in \{x\}$,[15] so that $x \in \big\{z \mid z \in x \lor z \in \{x\}\big\}$ by definition.[16] We can then clearly see that $x \in x \cup \{x\}$ by definition.[17] However, we know that $x \notin \varnothing$ because $\forall y\big(y \notin \varnothing\big)$. Therefore, the *axiom of extensionality* reveals $x \cup \{x\} \neq \varnothing$.

QED

[14] This is because $x = x$; *i.e.,* the *reflexivity of equality.*

[15] *disjunction introduction*

[16] *set comprehension notation*

[17] $x \cup y = \{z \mid z \in x \lor z \in y\}$

b. If we are restricted to only Axioms 0, 1, and 2, we can neither prove nor disprove the statement $\forall x \forall y\big(x \neq y \;\Rightarrow\; x \cup \{x\} \neq y \cup \{y\}\big)$.[18] To see why, let's try to ***attempt*** to prove this statement with the tools we have available to us.

[18] We will be able to prove this important fact once we have the *axiom of regularity.*

*"**Proof.**"* Let x and y be sets and assume $x \neq y$. Towards a contradiction, further assume $x \cup \{x\} = y \cup \{y\}$. Notice that $x \in \{x\}$ by definition and observe.

$$\begin{aligned} x \in \{x\} &\Rightarrow x \in x \lor x \in \{x\} && \text{by } \textit{disjunction introduction} \\ &\Rightarrow x \in \big\{z \mid z \in x \lor z \in \{x\}\big\} && \text{by definition} \\ &\Rightarrow x \in x \cup \{x\} && \text{by definition} \\ &\Rightarrow x \in y \cup \{y\} && \text{by the } \textit{axiom of extensionality} \\ &\Rightarrow x \in \Big\{z \;\Big|\; z \in y \lor z \in \{y\}\Big\} && \text{by definition} \\ &\Rightarrow x \in y \lor x \in \{y\} && \text{by definition} \\ &\Rightarrow x \in y \lor x = y && \text{by definition} \\ &\Rightarrow x \in y && \text{because } x \neq y \end{aligned}$$

This sequence of implications forces us to infer that $x \in y$. If we also notice that $y \in \{y\}$ by definition, we are compelled to draw a similar conclusion about y.

$$
\begin{aligned}
y \in \{y\} &\Rightarrow y \in y \lor y \in \{y\} && \text{by } \textit{disjunction introduction} \\
&\Rightarrow y \in \left\{z \,\middle|\, z \in y \lor z \in \{y\}\right\} && \text{by definition} \\
&\Rightarrow y \in y \cup \{y\} && \text{by definition} \\
&\Rightarrow y \in x \cup \{x\} && \text{by the } \textit{axiom of extensionality} \\
&\Rightarrow y \in \left\{z \,\middle|\, z \in x \lor z \in \{x\}\right\} && \text{by definition} \\
&\Rightarrow y \in x \lor x \in \{x\} && \text{by definition} \\
&\Rightarrow y \in x \lor y = x && \text{by definition} \\
&\Rightarrow y \in x && \text{because } x \neq y
\end{aligned}
$$

Therefore, we must necessarily have both $x \in y$ and $y \in x$.[19] At this point in our proof, we would be ***stuck***. Without more axioms, there is no contradiction we could draw from this seemingly strange relationship between x and y.

[19] We will soon use the *axiom of regularity* to disprove the existence of such sets.

However, suppose there existed two sets x and y such that $x = \{y\}$ and $y = \{x\}$[20] and $x \neq y$. Then, we could explicitly compute what $x \cup \{x\}$ and $y \cup \{y\}$ are.

[20] Using just Axioms 0, 1, and 2, we can not prove that such sets exist, but we also can't prove they *don't* exist.

$$
\begin{aligned}
x \cup \{x\} &= \left\{z \,\middle|\, z \in x \lor z \in \{x\}\right\} && \text{by definition of } \cup \\
&= \left\{z \,\middle|\, z \in \{y\} \lor z \in \{x\}\right\} && \text{because } x = \{y\} \\
&= \left\{z \,\middle|\, z = y \lor z = x\right\} && \text{by definition of } z \in \{y\} \text{ and } z \in \{x\} \\
&= \{y, x\} && \text{by the } \textit{axiom of extensionality} \\
y \cup \{y\} &= \left\{z \,\middle|\, z \in y \lor z \in \{y\}\right\} && \text{by definition of } \cup \\
&= \left\{z \,\middle|\, z \in \{x\} \lor z \in \{y\}\right\} && \text{because } y = \{x\} \\
&= \left\{z \,\middle|\, z = x \lor z = y\right\} && \text{by definition of } z \in \{x\} \text{ and } z \in \{y\} \\
&= \{x, y\} && \text{by the } \textit{axiom of extensionality}
\end{aligned}
$$

This clearly means $x \cup \{x\} = y \cup \{y\}$ despite $x \neq y$.