Problem Set 3

Discrete Structures

Due on the 15th day of February of the year of our Lord 2026 at 11:59 pm

As always, you may rely on any statement we have previously proven in lectures and problem sets. You should solve the problems *in order*; solutions to previous problems may be used as theorems to solve later problems.

The axioms, rules of inference, theorems, and other results of the zeroth-order and first-order logic can be used ***without explicitly citing them*** from this problem set onwards.

You should only need to rely on Axioms 0, 1, and 2 for this problem set.[1]

[1]From now on, when we refer to "Axioms" in the context of mathematics, we are referring to the axioms of Zermelo-Fraenkel set theory as we've presented them unless otherwise stated.

For any sets A and B, we make the following definitions:

- We say that $X \subseteq Y$ $:\Leftrightarrow$ $\forall z(z \in X \Rightarrow z \in Y)$.
- The *union over X* is given by $\cup X := \left\{y \,\middle|\, \exists x\big(x \in X \wedge y \in x\big)\right\}$.
- The *union of X with Y* is given by $X \cup Y := \{z \mid z \in X \vee z \in Y\}$.
- The *intersection over X* is given by $\cap X := \left\{y \,\middle|\, \forall x\big(x \in X \Rightarrow y \in x\big)\right\}$.
- The *intersection of X with Y* is given by $X \cap Y := \{z \mid z \in X \wedge z \in Y\}$.
- The *ordered pair* (x, y) is given by $(x, y) := \big\{\{x\}, \{x, y\}\big\}$.

1. a. Prove $\forall x \forall y \forall z\big((x \subseteq y \wedge y \subseteq z) \Rightarrow x \subseteq z\big)$.

   b. Prove $\forall x \forall y(x \cap y \subseteq x)$.

   c. Prove $\forall x \forall y(x \subseteq x \cup y)$.

   d. Prove $\forall x \forall y(x \cap y \subseteq x \cup y)$.

2. a. Prove $\forall x \forall y \exists p\big(p = (x, y)\big)$.

   b. Prove $\forall a \forall b \forall x \forall y\big((a, b) = (x, y) \Leftrightarrow (a = x \wedge b = y)\big)$.

3. a. Prove that $\forall x(x \cup \{x\} \neq \varnothing)$.

   b. Prove that $\forall x \forall y\big(x \neq y \Rightarrow x \cup \{x\} \neq y \cup \{y\}\big)$.