Solution Set 2

Discrete Structures

8th day of February of the year of our Lord 2026

As always, you may rely on any statement we have previously proven in lectures and problem sets. You should solve the problems *in order;* solutions to previous problems may be used as theorems to solve later problems.

1. We will prove each of the following statements for all propositions φ, ψ, ξ, and χ.

   a. We provide two proofs of $\vdash (\neg\varphi \to \varphi) \to \varphi$.

      Proof 1. Let φ be a proposition.

$$
\begin{aligned}
(\neg\varphi \to \varphi) \to \varphi &\equiv (\neg\neg\varphi \vee \varphi) \to \varphi && \text{by } \textit{conditional disintegration} \\
&\equiv (\varphi \vee \varphi) \to \varphi && \text{by } \textit{double negation} \\
&\equiv \varphi \to \varphi && \text{by } \textit{idempotency}
\end{aligned}
$$

   We know $\varphi \to \varphi$ by *problem 2.b.* from *Problem Set 1*. Therefore, $(\neg\varphi \to \varphi) \to \varphi$.

   <div align="right">QED</div>

   Proof 2. Let φ be a proposition. In the interest of setting up a use of the *deduction rule,* assume $\neg\varphi \to \varphi$. This is equivalent to $\neg\neg\varphi \vee \varphi$ by *conditional disintegration.* From this, we can derive $\varphi \vee \varphi$ by *double negation,* and then get φ by *idempotency.*

   Thus, we know $\neg\varphi \to \varphi \vdash \varphi$. Therefore, by the *deduction rule,* we can conclude $(\neg\varphi \to \varphi) \to \varphi$ as desired.

   <div align="right">QED</div>

   b. We provide two proofs of $\varphi \vdash \varphi \vee \psi$.

      Proof 1. Let φ and ψ be propositions. Assume φ. Observe the following.

$$
\begin{aligned}
\varphi \to (\varphi \vee \psi) &\equiv \neg\varphi \vee (\varphi \vee \psi) && \text{by } \textit{conditional disintegration} \\
&\equiv (\neg\varphi \vee \varphi) \vee \psi && \text{by } \textit{associativity} \\
&\equiv \top \vee \psi && \text{by } \textit{complement} \\
&\equiv \top && \text{by } \textit{domination}
\end{aligned}
$$

   We know $\top$ by *problem 2.c.* from *Problem Set 1*. Thus, we can say $\varphi \to (\varphi \vee \psi)$. Then, by *modus ponens,* we can conclude $\varphi \vee \psi$.

   <div align="right">QED</div>

   Proof 2. Let φ and ψ be propositions. Assume φ. Towards a contradiction, assume $\neg(\varphi \vee \psi)$, which is equivalent to $\neg\varphi \wedge \neg\psi$ by *De Morgan's laws.* We can then apply *conjunction elimination* to obtain $\neg\varphi$, clearly contradicting our assumption φ. ⚡

   Therefore, we can conclude $\varphi \vee \psi$ by *reductio ad absurdum.*

   <div align="right">QED</div>

c. We provide two proofs of $(\varphi \vee \psi), (\varphi \to \xi), (\psi \to \xi) \vdash \xi$.

Proof 1. Let φ, ψ, and ξ be propositions. Assume $\varphi \to \xi$ and $\psi \to \xi$. Also assume $\varphi \vee \psi$. By *conjunction introduction*, we then know $(\varphi \to \xi) \wedge (\psi \to \xi)$. Recall that $(\varphi \to \xi) \wedge (\psi \to \xi) \equiv (\varphi \vee \psi) \to \xi$ by *problem 1.e.* on *Problem Set 1*, producing $(\varphi \vee \psi) \to \xi$. We then have ξ by *modus ponens* as desired.

QED

Proof 2. Let φ, ψ, and ξ be propositions. Assume $\varphi \to \xi$ and $\psi \to \xi$. Also assume $\varphi \vee \psi$. Towards a contradiction, suppose $\neg \xi$.

We then know $\neg \varphi$ by *modus tollens*. Similarly, we have $\neg \psi$ by *modus tollens*. By *conjunction introduction*, we then have $\neg \varphi \wedge \neg \psi$. This is evidently equivalent to $\neg(\varphi \vee \psi)$ by *De Morgan's laws*, contradicting our assumption $\varphi \vee \psi$. ⨳

Therefore, by *reductio ad absurdum*, we can conclude ξ.

QED

d. We provide two proofs of $\varphi, \neg \varphi \vdash \psi$.

Proof 1. Let φ be a proposition. Assume φ, and also assume $\neg \varphi$.

Let ψ be an arbitrary proposition, and notice that $\varphi \to (\neg \psi \to \varphi)$ follows from *Hilbert's first axiom.*[1] Applying *modus ponens*, we then know $\neg \psi \to \varphi$. We can then see the following.

$$\neg \psi \to \varphi \equiv \neg \varphi \to \neg \neg \psi \qquad \text{by } \textit{problem 1.b. from Problem Set 1}$$
$$\equiv \neg \varphi \to \psi \qquad \text{by } \textit{double negation}$$

Thus, we have $\neg \varphi \to \psi$. Therefore, we conclude ψ by *modus ponens*.

QED

Proof 2. Let φ and ψ be propositions. Assume both φ and $\neg \varphi$. Towards a contradiction, suppose that $\neg \psi$. Recall that φ and also $\neg \varphi$. ⨳

Therefore, since $\neg \psi \vdash \varphi$ and $\neg \psi \vdash \neg \varphi$, we conclude ψ by *reductio ad absurdum*.

QED

e. We will prove $(\varphi \vee \psi), \neg \varphi \vdash \psi$.

Proof. Let φ and ψ be propositions. Assume $\varphi \vee \psi$, and assume $\neg \varphi$. Notice that $\varphi \vee \psi$ is equivalent to $\neg \neg \varphi \vee \psi$ by *double negation*, and this expression is equivalent to $\neg \varphi \to \psi$ by *conditional disintegration*. By *modus ponens*, we have ψ.

QED

f. We provide two proofs of $(\varphi \to \xi), (\psi \to \chi), (\varphi \vee \psi) \vdash (\xi \vee \chi)$.

Proof 1. Let φ, ψ, ξ, and χ be propositions. Assume $\varphi \to \xi$, $\psi \to \chi$, and $\varphi \vee \psi$. Notice that $\varphi \vee \psi$ is itself equivalent to $\neg \neg \varphi \vee \psi$ by *double negation*, and that is equivalent to $\neg \varphi \to \psi$ by *conditional disintegration*.

We have $\neg \varphi \to \chi$ by the *hypothetical syllogism*. This is equivalent to $\neg \chi \to \neg \neg \varphi$ by *problem 1.b.* from *Problem Set 1*. With the same justification, we know that our premise $\varphi \to \xi$ is equivalent to $\neg \xi \to \neg \varphi$. Applying the *hypothetical syllogism* again, we obtain $\neg \xi \to \chi$. This is equivalent to $\xi \vee \chi$ by *conditional disintegration*.

QED

[1] This is *Theorem 1.7* in the lecture notes.

Proof 2. Let φ, ψ, ξ, and χ be propositions. Assume $\varphi \to \xi$, $\psi \to \chi$, and $\varphi \lor \psi$.

Towards a contradiction, assume that $\neg(\xi \lor \chi)$, which we know is equivalent to $\neg\xi \land \neg\chi$ by *De Morgan's laws.* Notice that this is also equivalent to $\neg\chi \land \neg\xi$ by *commutativity.* We can thus use *conjunction elimination* to produce $\neg\xi$ and $\neg\chi$.

Now, from $\neg\xi$ and $\varphi \to \xi$, we obtain $\neg\varphi$ using *modus tollens.* Similarly, from $\neg\chi$ and $\psi \to \chi$, we also obtain $\neg\psi$ by *modus tollens.* We then have $\neg\varphi \land \neg\psi$ by *conjunction introduction.* This is equivalent to $\neg(\varphi \lor \psi)$ by *De Morgan's laws,* contradicting our earlier assumption $\varphi \lor \psi$. ⨯

Therefore, we can conclude $\xi \lor \chi$ by *reductio ad absurdum.*

<div align="right">QED</div>

2. a. Consider a universe of discourse containing all human beings who have ever existed or been mentioned in literature. Let μ and ω be unary predicates.

   We will prove $\forall x\big(\mu(x) \to \omega(x)\big), \mu(\text{Socrates}) \vdash \omega(\text{Socrates})$.

   Proof. Assume $\forall x\big(\mu(x) \to \omega(x)\big)$ and also $\mu(\text{Socrates})$. By *universal elimination,*[2] we have $\mu(\text{Socrates}) \to \omega(\text{Socrates})$. Therefore, $\omega(\text{Socrates})$ by *modus ponens.*

   [2] ... with the term *Socrates...*

   <div align="right">QED</div>

   b. Consider a non-empty universe of discourse consisting of all persons inhabiting New Port City, Japan in the year 2029 AD. Let α and γ be unary predicates.

   We will prove $\forall x\big(\neg\gamma(x) \to \alpha(x)\big), \neg\alpha(\text{Kusanagi}) \vdash \exists x\big(\gamma(x)\big)$.

   Proof. Assume $\forall x\big(\neg\gamma(x) \to \alpha(x)\big)$ and $\neg\alpha(\text{Kusanagi})$. By *universal elimination,*[3] we then know $\neg\gamma(\text{Kusanagi}) \to \alpha(\text{Kusanagi})$. By *modus tollens,* this produces $\neg\neg\gamma(\text{Kusanagi})$, which is equivalent to $\gamma(\text{Kusanagi})$ by *double negation.*

   [3] ... with the term *Kusanagi...*

   We can then conclude $\exists x\big(\gamma(x)\big)$ by *existential introduction.*

   <div align="right">QED</div>

   c. Let $\mathfrak{L}$ be a binary predicate. We demonstrate $\vdash \neg\exists x\forall y\big(\mathfrak{L}(x,y) \leftrightarrow \neg\mathfrak{L}(y,y)\big)$.

   Proof. Towards a contradiction, assume $\neg\neg\exists x\forall y\big(\mathfrak{L}(x,y) \leftrightarrow \neg\mathfrak{L}(y,y)\big)$, which is clearly equivalent to $\exists x\forall y\big(\mathfrak{L}(x,y) \leftrightarrow \neg\mathfrak{L}(y,y)\big)$ by *double negation.*

   By *existential elimination,* we then know $\forall y\big(\mathfrak{L}(t,y) \leftrightarrow \neg\mathfrak{L}(y,y)\big)$ for some t. We can then apply *universal elimination*[4] to assert $\mathfrak{L}(t,t) \leftrightarrow \neg\mathfrak{L}(t,t)$. Now, observe.

   [4] ... with the term *t...*

$$
\begin{aligned}
\mathfrak{L}(t,t) \leftrightarrow \neg\mathfrak{L}(t,t) &\equiv \big(\mathfrak{L}(t,t) \to \neg\mathfrak{L}(t,t)\big) \land \big(\neg\mathfrak{L}(t,t) \to \mathfrak{L}(t,t)\big) && \text{by } \textit{bicond. dis.} \\
&\equiv \big(\neg\mathfrak{L}(t,t) \lor \neg\mathfrak{L}(t,t)\big) \land \big(\neg\neg\mathfrak{L}(t,t) \lor \mathfrak{L}(t,t)\big) && \text{by } \textit{cond. dis.} \\
&\equiv \big(\neg\mathfrak{L}(t,t) \lor \neg\mathfrak{L}(t,t)\big) \land \big(\mathfrak{L}(t,t) \lor \mathfrak{L}(t,t)\big) && \text{by } \textit{double neg.} \\
&\equiv \neg\mathfrak{L}(t,t) \land \mathfrak{L}(t,t) && \text{by } \textit{idemp.} \\
&\equiv \bot && \text{by } \textit{compl.}
\end{aligned}
$$

   Therefore, $\bot$, which is equivalent to $\neg\top$ thanks to *Corollary 1.1* from the lecture notes. However, $\top$ by *problem 2.c.* from *Problem Set 1.* ⨯

   Therefore, $\neg\exists x\forall y\big(\mathfrak{L}(x,y) \leftrightarrow \neg\mathfrak{L}(y,y)\big)$ by *reductio ad absurdum.*

   <div align="right">QED</div>

3. Consider a universe of discourse consisting of the natural number and define the predicate $\pi(x) := $ "x is a prime number." Let φ be the statement $\varphi \to \forall x(\pi(x))$.[5]

We will prove that 4 is prime lmao.

Proof. The first thing we will do is prove φ. Recall that φ is defined to be the statement $\varphi \to \forall x(\pi(x))$, which is a *conditional* statement. In order to prove this, we will set up a use of the *deduction rule.*

Assume φ. By definition, this means we know $\varphi \to \forall x(\pi(x))$. Now, by applying *modus ponens,* we have $\forall x(\pi(x))$. Therefore, we know $\varphi \vdash \forall x(\pi(x))$ by definition. So, $\varphi \to \forall x(\pi(x))$ by the *deduction rule.* This means we have proven φ by definition.

Okay, now we know φ. By definition, this means we know $\varphi \to \forall x(\pi(x))$. Then, by *modus ponens,* we can say that $\forall x(\pi(x))$. As a result, by *universal elimination,* we assert $\pi(4)$, which says precisely that 4 is a prime number, concluding our proof.

<div align="right">QED</div>

[5] This is an example of a *Curry sentence,* named after mathematician and computer scientist Haskell Curry.