Problem Set 2

Discrete Structures

Due on the 8th day of February of the year of our Lord 2026 at 11:59 pm

As always, you may rely on any statement we have previously proven in lectures and problem sets. You should solve the problems *in order;* solutions to previous problems may be used as theorems to solve later problems.

1. Prove each of the following statements for all propositions φ, ψ, ξ, and χ.[1]

   a. $\vdash (\neg\varphi \to \varphi) \to \varphi$ — *Consequentia Mirabilis*

   b. $\varphi \vdash \varphi \vee \psi$ — *Disjunction Introduction*

   c. $(\varphi \vee \psi), (\varphi \to \xi), (\psi \to \xi) \vdash \xi$ — *Disjunction Elimination*

   d. $\varphi, \neg\varphi \vdash \psi$ — *Ex Falso Quodlibet*

   e. $(\varphi \vee \psi), \neg\varphi \vdash \psi$ — *Disjunctive Syllogism*

   f. $(\varphi \to \xi), (\psi \to \chi), (\varphi \vee \psi) \vdash (\xi \vee \chi)$ — *Constructive Dilemma*

2. Prove each of the following statements.[2]

   a. Consider a universe of discourse containing all human beings who have ever existed or been mentioned in literature. Let μ and ω be unary predicates.

   Prove that $\forall x\big(\mu(x) \to \omega(x)\big), \mu(\text{Socrates}) \vdash \omega(\text{Socrates})$.

   b. Consider a non-empty universe of discourse consisting of all persons inhabiting New Port City, Japan in the year 2029 AD. Let α and γ be unary predicates.

   Prove that $\forall x\big(\neg\gamma(x) \to \alpha(x)\big), \neg\alpha\big(\text{Kusanagi}\big) \vdash \exists x\big(\gamma(x)\big)$. — 草薙 素子

   c. Let $\mathfrak{L}$ be a binary predicate and show that $\vdash \neg\exists x\forall y\Big(\mathfrak{L}(x,y) \leftrightarrow \neg\mathfrak{L}(y,y)\Big)$.

3. For this problem, we will be treating a statement that *is not a proposition* as if it *is* a proposition as a cautionary tale. Remember that, for the purposes of proof-writing, we do not yet *know* any facts about the natural numbers.[3]

   Consider a universe of discourse consisting of the natural number[4] and define the predicate $\pi(x) :=$ "x is a prime number." Let φ be the statement $\varphi \to \forall x\big(\pi(x)\big)$.[5]

   Prove that 4 is a prime number.

[1] For problem 1, you may rely on everything *before page 26* of the lecture notes.

[2] For problem 2, you may rely on everything in *chapters 0, 1, and 2* of the lecture notes.

[3] For problem 3, you may rely on everything in *chapters 0, 1, and 2* of the lecture notes.

[4] Recall that the natural numbers consist of $0, 1, 2, 3, 4, \ldots$

[5] For the sake of this problem, you should assume that φ is a proposition, and therefore obeys all of the axioms, rules of inference, and theorems about propositions we have studied so far.