Solution Set 1

Discrete Structures

1st day of February of the year of our Lord 2026

1. a. We will prove $p \to p$ is a tautology for any proposition p.

   Proof. Let p be a proposition. Observe the following.

$$\begin{aligned} p \to p &\equiv \neg p \lor p && \text{by } \textit{conditional disintegration} \\ &\equiv \top && \text{by } \textit{complement} \end{aligned}$$

   Therefore, $p \to p \equiv \top$, which means $p \to p$ is a tautology by definition.

   QED

   b. We will prove $p \to q \equiv \neg q \to \neg p$ for all propositions p and q.

   Proof. Let p and q be propositions. Bear witness to the reasoning below.

$$\begin{aligned} p \to q &\equiv \neg p \lor q && \text{by } \textit{conditional disintegration} \\ &\equiv q \lor \neg p && \text{by } \textit{commutativity} \\ &\equiv \neg\neg q \lor \neg p && \text{by } \textit{double negation} \\ &\equiv \neg q \to \neg p && \text{by } \textit{conditional disintegration} \end{aligned}$$

   Therefore, we have $p \to q \equiv \neg q \to \neg p$ as desired.

   QED

   c. We will prove $\neg(p \to q) \equiv p \land \neg q$ for all propositions p and q.

   Proof. Let p and q be propositions. We should now turn our gaze below.

$$\begin{aligned} \neg(p \to q) &\equiv \neg(\neg p \lor q) && \text{by } \textit{conditional disintegration} \\ &\equiv \neg\neg p \land \neg q && \text{by } \textit{De Morgan's laws} \\ &\equiv p \land \neg q && \text{by } \textit{double negation} \end{aligned}$$

   Therefore, we can conclude $\neg(p \to q) \equiv p \land \neg q$.

   QED

   d. We show $\big(p \land (p \to q)\big) \to q$ is a tautology for all propositions p and q.

   Proof. Let p and q be propositions. We are now compelled to observe.

$$\begin{aligned} \big(p \land (p \to q)\big) \to q &\equiv \neg\big(p \land (\neg p \lor q)\big) \lor q && \text{by } \textit{conditional disintr.} \\ &\equiv \big(\neg p \lor \neg(\neg p \lor q)\big) \lor q && \text{by } \textit{De Morgan's laws} \\ &\equiv \big(\neg p \lor (\neg\neg p \land \neg q)\big) \lor q && \text{by } \textit{De Morgan's laws} \\ &\equiv \big(\neg p \lor (p \land \neg q)\big) \lor q && \text{by } \textit{double negation} \\ &\equiv \big((\neg p \lor p) \land (\neg p \lor \neg q)\big) \lor q && \text{by } \textit{distributivity} \\ &\equiv \big(\top \land (\neg p \lor \neg q)\big) \lor q && \text{by } \textit{complement} \\ &\equiv (\neg p \lor \neg q) \lor q && \text{by } \textit{identity} \\ &\equiv \neg p \lor (\neg q \lor q) && \text{by } \textit{associativity} \end{aligned}$$

$$\begin{aligned} &\equiv \neg p \lor \top && \text{by } \textit{complement} \\ &\equiv \top \lor \neg p && \text{by } \textit{commutativity} \\ &\equiv \top && \text{by } \textit{domination} \end{aligned}$$

Therefore, we are forced to admit that $\big(p \land (p \to q)\big) \to q$ is a tautology.

QED

e. We prove $(p \lor q) \to r \equiv (p \to r) \land (q \to r)$ for all propositions p, q, and r.

Proof. Let p, q, and r be propositions. Look.

$$\begin{aligned} (p \lor q) \to r &\equiv \neg(p \lor q) \lor r && \text{by } \textit{conditional disintegration} \\ &\equiv (\neg p \land \neg q) \lor r && \text{by } \textit{De Morgan's laws} \\ &\equiv r \lor (\neg p \land \neg q) && \text{by } \textit{commutativity} \\ &\equiv (r \lor \neg p) \land (r \lor \neg q) && \text{by } \textit{distributivity} \\ &\equiv (\neg p \lor r) \land (\neg q \lor r) && \text{by } \textit{commutativity} \\ &\equiv (p \to r) \land (q \to r) && \text{by } \textit{conditional disintegration} \end{aligned}$$

As such, $(p \lor q) \to r \equiv (p \to r) \land (q \to r)$ as desired.

QED

2. a. We will show $(\varphi \to \psi), (\psi \to \xi) \vdash \varphi \to \xi$ for all propositions φ, ψ, and ξ.

Proof. Let φ, ψ, and ξ be propositions. Assume $\varphi \to \psi$ and $\psi \to \xi$.

Assume φ. Recall that $\varphi \to \psi$. We can then derive ψ by *modus ponens*. Because we know $\psi \to \xi$, we can again apply *modus ponens* to obtain ξ.

Since we were able to prove ξ by assuming φ, we can say that $\varphi \vdash \xi$. Therefore, by the *deduction rule*, we conclude $\varphi \to \xi$.

QED

We would like to prove that $\varphi \to \xi$. In order to do this, we will set up a use of the *deduction rule* by first proving that $\varphi \vdash \xi$.

b. We will prove $\vdash \varphi \to \varphi$ for any proposition φ.

Proof. Let φ be a proposition. Assume φ. Lo, behold, we now know φ. Thus, we have $\varphi \vdash \varphi$. The *deduction rule* therefore produces $\varphi \to \varphi$.

QED

In order to prove $\varphi \to \varphi$, we will set up a use of the *deduction rule* by first proving $\varphi \vdash \varphi$. In order to show $\varphi \vdash \varphi$, we need to assume φ and then derive φ as a consequence.

c. We will prove $\vdash \top$.

Proof. From *problem 2.b.*, we know that $\top \to \top$. Notice that $\top \to \top \equiv \top$ from *problem 1.a.* Therefore, we have $\top$ as desired.

QED

d. We will prove $\varphi, \psi \vdash \varphi \land \psi$ for all propositions φ and ψ.

Proof. Let φ and ψ be propositions. Assume φ, and also assume ψ.

Towards a contradiction, suppose $\neg(\varphi \land \psi)$. Observe the following.

$$\begin{aligned} \neg(\varphi \land \psi) &\equiv \neg\varphi \lor \neg\psi && \text{by } \textit{De Morgan's laws} \\ &\equiv \varphi \to \neg\psi && \text{by } \textit{conditional disintegration} \end{aligned}$$

We thus have $\varphi \to \neg\psi$. Recalling φ, we obtain $\neg\psi$ by *modus ponens*. However, this contradicts ψ. ↯

Since $\neg(\varphi \wedge \psi) \vdash \psi$ and $\neg(\varphi \wedge \psi) \vdash \neg\psi$, we can apply *reductio ad absurdum* to conclude $\varphi \wedge \psi$.

QED

e. We will prove $\varphi \wedge \psi \vdash \varphi$ for all propositions φ and ψ.

Proof. Let φ and ψ be propositions, and assume $\varphi \wedge \psi$.

Towards a contradiction, assume $\neg\varphi$. Recalling that $\varphi \wedge \psi$, we can apply *Conjunction Introduction*[1] to obtain $\neg\varphi \wedge (\varphi \wedge \psi)$. Now, observe the following deduction.

[1]This is the name of the theorem that was proven in *problem 2.d.*

$$\begin{aligned} \neg\varphi \wedge (\varphi \wedge \psi) &\equiv (\neg\varphi \wedge \varphi) \wedge \psi && \text{by } \textit{associativity} \\ &\equiv \bot \wedge \psi && \text{by } \textit{complement} \\ &\equiv \bot && \text{by } \textit{domination} \end{aligned}$$

Thus, we now know $\bot$. Since we know $\bot \equiv \neg\top$,[2] we can now say $\neg\top$. However, this contradicts $\top$, which we get from *problem 2.c.* ↯

[2]This theorem was proven during lecture.

Therefore, since we saw that $\neg\varphi \vdash \top$ and $\neg\varphi \vdash \neg\top$, we can apply *reductio ad absurdum* to conclude φ as desired.

QED