Solution Set 0

Discrete Structures

25th day of January of the year of our Lord 2026

1. We will determine the truth value of each of the following sentences.

   a. "Every integer is either even or odd." is *true*.

   We will call a number z "even" if z is an integer multiple of 2, and we will call z "odd" if it is one more than a multiple of 2.

   Using this intuition, we can see that 0 is even because $0 = 0 \cdot 2$. Since $2x + 2 = 2(x + 1)$, we obtain all of the positive even integers by starting at 0 and continually adding 2. Because $2x - 2 = 2(x - 1)$, we obtain the negative even integers by starting at 0 and continually adding -2.

   The only numbers left over after this process must lie between two even integers. However, after every even integer $2x$, there is an odd[1] integer $2x + 1$, and the next number after that $2x + 2$, which equals $2(x + 1)$, is even. So, these numbers that lie between the even integers are precisely the odd integers. Therefore, every integer is either even or odd.

   [1]This follows from our definition of "odd."

   b. "Every number is either even or odd." is *false*.

   Consider the number $^1/_2$. To see that there is no way to write this as a multiple of 2, suppose that we *could*. That would mean $^1/_2 = 2x$ for some integer x, and therefore $x = {}^1/_4$. We can see that $0 < x < 1$, which is absurd because there are no integers between 0 and 1.

   Therefore, $^1/_2$ is not even. By repeating a similar argument, we can see that $^{-1}/_2$ is also not even. This shows $^1/_2$ is also not odd, since the number it is one greater than is not even.

   c. "Every non-empty set of integers has a least element." is *false*.

   To see that this sentence is false, we will find an example of a non-empty set of integers with no least element. Consider, for example, the set of all even integers. To see that this set has no least element, take an arbitrary even integer z and notice that $z - 2$ is also even and is less than z. This argument generalizes to any even integer, so this set has no least element.

   d. "Every non-empty set of natural numbers has a least element." is *true*.

   Recall that 0 is the smallest natural number. Consider an arbitrary non-empty set of natural numbers A and let a_0 be one of its elements.[2]
   If a_0 is not the least element of A, then there must be an element a_1 such that $a_1 \leqslant a_0 - 1 < a_0$. Now, if a_1 is not the least element of A, there must be an element a_2 such that $a_2 \leqslant a_1 - 1 < a_1$. We can continue asking this question, looking for the least element of A, and after at most a_0 iterations of this process we will exhaust all of the numbers between 0 and a_0. Since there are no natural numbers less than 0, one of the numbers between 0 and a_0 must be the least element of A.

   [2]Notice that we know there exists such an element a_0 in the first place because A is non-empty.

e. "Every natural number is boring." is *false*.

For example, 0 has the interesting property that $0 \leqslant x$ for any natural number x. This precludes the natural number 0 from being boring.

f. "No natural number is boring." is *true*.

To show that every natural number is interesting, suppose towards the contrary that there is at least one boring number. Now, consider the set of all boring numbers B. This is a non-empty set of natural numbers, so it has a least element b. This number b is the *smallest* boring number… a very interesting property! This means b is *not* boring, contradicting the fact that b is the smallest *boring* number. Therefore, our assumption that there exists at least one boring number was mistaken, and there are no boring natural numbers.

2. We will determine the truth value of each of the following sentences.

a. "This sentence is *true*." could have *either truth value*.

I mean, literally just read it. If you believe it, then the sentence is certainly *true*. However, if you don't believe it, then you would say that the sentence is *false* and you would believe the opposite of what it says: that the sentence is *false*. In either case, there's no problem with assigning either truth value to the sentence.

b. "This sentence is *false*." is *neither truth nor false*.

Let S be the sentence "S is *false*" and suppose S is *true*. Then, we should believe what S says, and therefore we believe S is *false*, immediately contradicting our assumption.

On the other hand, assuming S is *false*, we should then believe the opposite of what S says. This means we believe S is not *false*, again contradicting our assumption.

Since we run into contradictions in either case, there is no consistent truth value that can be assigned to that sentence.

c. "The set of all sets that contain themselves contains itself." could have *either truth value*.

We should first suppose that such an object like "the set of all sets that contain themselves" exists and that we can talk about it in a consistent way. Let's give this set the name B. The elements of B are all and only those sets x such that x contains x as an element.

If the given sentence is *true*, then we should believe that U contains itself as an element. By the definition of U, that means that U is an element of U, which agrees with the observation we just made. It seems that no contradictions are introduced.

If we instead suppose that the given sentence is *false*, then we would *not* believe that U contains itself. By the definition of U, we then know U is *not* one of the sets that contains itself, meaning that U does not contain itself. This agrees with our observation that the given sentence is *false*, and again it seems like all is right with the world.

Therefore, much like the sentence in 2.a., this sentence could consistently hold either truth value.

d. "The set of all sets that do not contain themselves does not contain itself." is *neither true nor false*.

Suppose that such a set exists, and call it R. If the sentence is *true,* then R contains R as an element, and therefore R is one of the sets that does not contain itself, contradicting the fact that R contains R.

Alternatively, suppose the sentence is *false.* Then, R is *not* an element of R, so R is not one of the sets that contains themselves. However, since R does not contain itself, it qualifies as one of the elements of R by definition, contradicting our previous observation.

e. "If this sentence is *false,* then 7 is a prime number." is *true.*

Call the sentence above S, so that it says "If S is *false,* then 7 is prime." If S were *false,* this would imply that 7 is prime. But 7 is actually prime, so the inference that "if S is *false,* then 7 is prime" is *true,* meaning that S is *true,* contradicting the fact that we assumed S was *false.*

If we simply say that S is *true,* then S is a conditional statement with a *false* premise, which is *true* by default, and all is right with the world.

f. "If this sentence is *true,* then 7 is not a prime number." is *neither true nor false.*

Let C be the sentence above. C says "If C is *true,* then 7 is not prime."

First, assume C is *false.* That means C is an implication with a *false* premise and *true* conclusion. However, this contradicts the fact that we know 7 *is* prime.

Second, assume C is *true.* This means the premise of C is *true,* and thus we know the conclusion of C must be *true.* This tells us 7 is not prime, which again contradicts the fact that 7 is prime.

Therefore, neither truth value can consistently be assigned to C.

3. Consider the infinite sequence of sentences indexed by natural numbers

$$S_0, S_1, S_2, \ldots, S_i, \ldots$$

in which each sentence asserts that every sentence following it is *false.*[3]

$$S_i := \text{“}S_j \text{ is } \textit{false} \text{ for all } j > i\text{”}$$

[3]The variable i in the definition of S_i ranges over all of the natural numbers $0, 1, 2, \ldots$

We will show that *none of these sentences has a consistent truth value.*

Consider the first sentence S_0 and suppose S_0 is *true.* We then know that all of the sentences $S_1, S_2, S_3, \ldots$ are *false.* In particular, this means that S_1 is *false,* which implies that *not all* of the sentences $S_2, S_3, S_4, \ldots$ are *false.* That means that there is some sentence S_i such that S_i is *true,* where $i > 1$. However, according to S_0, we know that S_i is *false* because $i > 1$. ↯

Let's then analyze what would happen if S_0 were *false.* That would mean that not all of the sentences $S_1, S_2, S_3, \ldots$ are *false,* so there must be a sentence S_k such that S_k is *true,* where $k > 0$. Following similar logic as before, this implies S_{k+1} is *false,* which means there is some sentence S_i such that S_i is *true,* where $i > k + 1$. However, S_i being *true* with $i > k + 1 > k$ contradicts the fact that we said S_k was *true.* ↯

Thus, there is no consistent way to assign a truth value to S_0. We can take this argument about S_0 and generalize it to any sentence in this sequence, and therefore none of them have consistent truth values.