NAME: NETID:

Practice Final

Discrete Structures

9^{th} day of May of the year of our Lord 2026

1 *Answer each of the following questions by marking True or False but not both.*

1. If a and b are odd, then $(\forall c \in \mathbb{Z})\left(c^2 = a^2 + b^2\right)$.

   ○ *True*
   ○ *False*

2. If this statement has a truth value, then every countable set is finite.

   ○ *True*
   ○ *False*

3. Every graph must contain an even number of even-degree vertices.

   ○ *True*
   ○ *False*

4. There exist $x, y \in \mathbb{Z} \setminus \{0\}$ such that $5 \mid x$ and $5 \mid y$ and $\gcd(x, y) < 5$.

   ○ *True*
   ○ *False*

5. $\forall x \forall y \left(x \subseteq y \Leftrightarrow \mathbb{P}(X) \subseteq \mathbb{P}(y)\right)$.

   ○ *True*
   ○ *False*

6. For any set A, there exists $B \subseteq A$ such that $|A \setminus B| = |A|$.

   ○ *True*
   ○ *False*

7. There are as many propositions as there are natural numbers.

   ○ *True*
   ○ *False*

8. If $n \in \mathbb{N}$ and a graph G has n nodes and $n - 1$ edges, then G is connected.

   ○ *True*
   ○ *False*

9. There exists $x \in \mathbb{Z}$ such that $16x \equiv 50 \pmod 7$.

   ○ *True*
   ○ *False*

10. There is a tree T such that $|V(T)|$ and $|E(T)|$ are both prime numbers greater than 2.

    ○ *True*
    ○ *False*

2 *Answer each of the following questions without proof.*

1. Let $T := \big\{t \;\big|\; (\exists n \in \mathbb{N})\big(t : n \to \{0,1,2\}\big)\big\}$. Provide an explicit bijection $\varphi : T \to \mathbb{N}$.

2. Give an example of an $\in$-transitive set that is not a natural number.

3. Give an example of an uncountable set X such that $\forall Y\big(X \neq \mathbb{P}(Y)\big)$.

4. Given a positive natural number $n \in \mathbb{N}_+$, what is the *minimum* number of edges a connected graph on n nodes can have?

5. Give an example of a connected graph G for which 3 is the *minimum* number of edges whose removal would disconnect G.

3 *The axioms, rules of inference, and theorems of the zeroth-order and first-order logic **must** be explicitly cited (by name, if possible) when used.*

Prove the statement below for any propositions p and q.

$$\vdash \ \big(\neg q \land (p \to q)\big) \to \neg p$$

4 *You may assume all axioms and theorems we have discussed so far, including any basic arithmetic and algebraic properties of* $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, *and* $\mathbb{R}$.

Use induction to prove that $(\forall n \in \mathbb{N})\big(n \geqslant 4 \;\Rightarrow\; (\exists x, y \in \mathbb{Z})(2x + 5y = n)\big)$.

5 *You may assume all axioms and theorems we have discussed so far, including any basic arithmetic and algebraic properties of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.*

Prove that there are infinitely many finite binary strings.

6 *You may assume all axioms and theorems we have discussed so far, including any basic arithmetic and algebraic properties of* $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, *and* $\mathbb{R}$.

Use induction to prove that any connected graph contains a spanning tree.