Solutions for Practice Midterm 0

Discrete Structures

23rd day of February of the year of our Lord 2026

1 *Answer each of the following questions by marking either True or False but not both. You may assume all of the axioms of Zermelo-Fraenkel set theory.*

1. If this sentence is *false*, then $\varnothing$ is empty.

   ● *True*
   ○ *False*

2. $\big\{\{\varnothing,\{\varnothing\}\},\{\varnothing,\{\varnothing\}\}\big\}$ is a natural number.

   ○ *True*
   ● *False*

3. $\mathbb{P}(\varnothing)$ is a subset of every set.

   ○ *True*
   ● *False*

4. $\forall x \exists y (y \in x)$.

   ○ *True*
   ● *False*

5. $\forall x \exists y (y \subseteq x)$.

   ● *True*
   ○ *False*

6. This sentence has no truth value.

   ○ *True*
   ● *False*

7. $\exists x \exists y \exists z \big((x \in y) \wedge (y \in z) \wedge (z \in x)\big)$.

   ○ *True*
   ● *False*

8. $\exists x \exists y \exists z \big((x \subseteq y) \wedge (y \subseteq z) \wedge (z \subseteq x)\big)$.

   ● *True*
   ○ *False*

9. $\Big\{x \;\Big|\; \exists y \big(y = \{z \mid \exists w (w \in x \wedge z \in w)\}\big)\Big\}$ exists.

   ○ *True*
   ● *False*

10. The sentence given in problem 6 on this page has no truth value.

    ○ *True*
    ● *False*

2 *The axioms, rules of inference, and theorems of the zeroth-order and first-order logic must be explicitly cited when used.*

For any propositions α, β, γ, prove the statement shown below.

$$\alpha \vee \beta,\ \beta \to \alpha,\ \alpha \to \gamma \ \vdash\ \gamma$$

Proof 1. Let α, β, and γ be propositions. Assume $\alpha \vee \beta$, and $\beta \to \alpha$, and also $\alpha \to \gamma$. Towards a contradiction, assume $\neg\gamma$. By *modus tollens,* we obtain $\neg\alpha$. By the *modus tollens* again, we further obtain $\neg\beta$. Thus, we can see $\neg\alpha \wedge \neg\beta$ by *conjunction introduction.*[1] Now, observe $\neg\alpha \wedge \neg\beta \equiv \neg(\alpha \vee \beta)$ by *De Morgan's laws,* producing $\neg(\alpha \vee \beta)$ and contradicting our earlier assumption $\alpha \vee \beta$. ↯

[1] $x, y \vdash x \wedge y$

Therefore, we have γ as desired.

QED

Proof 2. Let α, β, and γ be propositions. Assume $\alpha \vee \beta$, and also $\beta \to \alpha$, and $\alpha \to \gamma$ as well. By the *hypothetical syllogism,*[2] we know $\beta \to \gamma$. Now, observe the following.

[2] $x \to y,\ y \to z \vdash x \to z.$

$$\begin{aligned} \alpha \vee \beta &\equiv \neg\neg\alpha \vee \beta && \text{by } \textit{double negation} \\ &\equiv \neg\alpha \to \beta && \text{by } \textit{conditional disintegration} \\ &\equiv \neg\beta \to \neg\neg\alpha && \text{by a theorem we proved} \\ &\equiv \neg\beta \to \alpha && \text{by } \textit{double negation} \end{aligned}$$

So we know $\neg\beta \to \alpha$ now. Again, by the *hypothetical syllogism,* we can derive $\neg\beta \to \gamma$. Thanks to a theorem we have proven, we know $\top$. We also know $\top \equiv \neg\beta \vee \beta$ by *complement*, so we know $\neg\beta \vee \beta$.

Now, we have $\neg\beta \vee \beta$, and we know both $\beta \to \gamma$ and $\neg\beta \to \gamma$. Therefore, we can conclude γ by *disjunction elimination.*[3]

[3] $x \vee y,\ x \to z,\ y \to z \vdash z$

QED

3 *The axioms, rules of inference, and theorems of the zeroth-order and first-order logic **need not** be cited. You may assume axioms 0 through 6 of set theory, and you may rely on any theorems proven in lectures, notes, and problem sets.*

Show that $\forall x \forall y (x \cap y = \varnothing \;\Rightarrow\; x \subseteq x \setminus y)$.

Proof 1. Let x and y be arbitrary sets, and assume $x \cap y = \varnothing$. Let t be a set, and suppose that $t \in x$. Recall that $t \notin \varnothing$ because $\forall w (w \notin \varnothing)$. Observe the following derivation.

$$\begin{aligned} t \notin \varnothing \;&\Rightarrow\; t \notin x \cap y && \text{by the \textit{axiom of extensionality}} \\ &\Rightarrow\; \neg(t \in x \land t \in y) && \text{by definition} \\ &\Rightarrow\; t \notin x \lor t \notin y \end{aligned}$$

Since $t \in x$, this implies $t \notin y$, and so $t \in x \setminus y$ by definition. Therefore, $x \subseteq x \setminus y$.

QED

Proof 2. Let x and y be arbitrary sets, and assume $x \cap y = \varnothing$. Towards a contradiction, suppose $x \not\subseteq x \setminus y$. By definition, means there exists t such that $t \in x$ and $t \notin x \setminus y$. Observe the following deduction.

$$\begin{aligned} t \notin x \setminus y \;&\Rightarrow\; \neg(t \in x \land t \notin y) && \text{by definition} \\ &\Rightarrow\; t \notin x \lor t \in y \end{aligned}$$

Since $t \in x$, we can deduce that $t \in y$. Thus, $t \in x \cap y$ by definition, so $t \in \varnothing$ by the *axiom of extensionality*. However, $t \notin \varnothing$ because $\forall w (w \notin \varnothing)$. ↯

Therefore, we conclude $x \subseteq x \setminus y$ as desired.

QED

4 *The axioms, rules of inference, and theorems of the zeroth-order and first-order logic* ***need not*** *be cited. You may assume axioms 0 through 6 of set theory, and you may rely on any theorems proven in lectures, notes, and problem sets.*

Show that $\forall x \forall y \big(\mathbb{P}(x) \subseteq \mathbb{P}(y) \Rightarrow x \subseteq y \big)$.

Proof. Let x and y be sets, and assume $\mathbb{P}(x) \subseteq \mathbb{P}(y)$. Recall that $x \subseteq x$ because we have proven $\forall w(w \subseteq w)$. Thus, $x \in \mathbb{P}(x)$ by definition, so $x \in \mathbb{P}(y)$ thanks to our assumption. This means $x \subseteq y$ by definition.

QED