

Esoterica 1

Discrete Structures

4th day of February of the year of our Lord 2026

1. Fix a universe of discourse containing all living people, and consider following.¹

- Every mathematician likes baristas.
- Every barista knows how to make coffee.
- Dan is a mathematician.
- Anyone who likes baristas likes people who know how to make coffee.
- Therefore, someone likes people know how to make coffee.

¹Whether you decide to disambiguate this argument using \forall (as we did in class) or \exists , this argument is still valid (*i.e.*, the conclusion can be proven from the given premises).

We will formalize both ways of disambiguating this argument and prove both. Let us define the following predicates.

$$\mu(x) := \text{"}x \text{ is a mathematician."}$$

$$\beta(x) := \text{"}x \text{ is a barista."}$$

$$\kappa(x) := \text{"}x \text{ knows how to make coffee."}$$

$$\lambda(x, y) := \text{"}x \text{ likes } y\text{"}$$

a. First, consider the following formalization using universal quantifiers.

- $\forall x(\mu(x) \rightarrow \forall y(\beta(y) \rightarrow \lambda(x, y)))$.
- $\forall x(\beta(x) \rightarrow \kappa(x))$.
- $\mu(\text{Dan})$.
- $\forall x(\forall y(\beta(y) \rightarrow \lambda(x, y)) \rightarrow \forall y(\kappa(y) \rightarrow \lambda(x, y)))$.
- Therefore, $\exists x\forall y(\kappa(y) \rightarrow \lambda(x, y))$.

We will show that the last line is provable from the first four.

Proof. Assume $\forall x(\mu(x) \rightarrow \forall y(\beta(y) \rightarrow \lambda(x, y)))$. Also, assume $\forall x(\beta(x) \rightarrow \kappa(x))$ and $\mu(\text{Dan})$. Finally, assume $\forall x(\forall y(\beta(y) \rightarrow \lambda(x, y)) \rightarrow \forall y(\kappa(y) \rightarrow \lambda(x, y)))$.

We can derive $\mu(\text{Dan}) \rightarrow \forall y(\beta(y) \rightarrow \lambda(\text{Dan}, y))$ by *universal elimination*.

Using *modus ponens*, we can then see $\forall y(\beta(y) \rightarrow \lambda(\text{Dan}, y))$.

Now, let's look at our last assumption. By applying *universal elimination*, we can derive $\forall y(\beta(y) \rightarrow \lambda(\text{Dan}, y)) \rightarrow \forall y(\kappa(y) \rightarrow \lambda(\text{Dan}, y))$.

Taking this with our previous result above, we can apply *modus ponens* and obtain the expression $\forall y(\kappa(y) \rightarrow \lambda(\text{Dan}, y))$.

Since we have produced $\forall y(\kappa(y) \rightarrow \lambda(\text{Dan}, y))$ and "Dan" is a specific object, we can use *existential introduction* to conclude $\exists x\forall y(\kappa(y) \rightarrow \lambda(x, y))$. QED

b. Now, consider the following formalization using existential quantifiers.

- $\forall x(\mu(x) \rightarrow \exists y(\beta(y) \wedge \lambda(x, y)))$.
- $\forall x(\beta(x) \rightarrow \kappa(x))$.
- $\mu(\text{Dan})$.
- $\forall x(\exists y(\beta(y) \wedge \lambda(x, y)) \rightarrow \exists y(\kappa(y) \wedge \lambda(x, y)))$.
- Therefore, $\exists x\exists y(\beta(y) \wedge \lambda(x, y))$.

We will prove the last line above is also provable from the first four statements.

Proof. Assume $\forall x(\mu(x) \rightarrow \exists y(\beta(y) \wedge \lambda(x, y)))$. Also, assume $\forall x(\beta(x) \rightarrow \kappa(x))$ and $\mu(\text{Dan})$. Finally, assume $\forall x(\exists y(\beta(y) \wedge \lambda(x, y)) \rightarrow \exists y(\kappa(y) \wedge \lambda(x, y)))$.

We can derive $\mu(\text{Dan}) \rightarrow \exists y(\beta(y) \wedge \lambda(\text{Dan}, y))$ by *universal elimination*.

Using *modus ponens*, we can then see $\exists y(\beta(y) \wedge \lambda(\text{Dan}, y))$.

Now, let's look at our last assumption. By applying *universal elimination*, we can derive $\exists y(\beta(y) \wedge \lambda(\text{Dan}, y)) \rightarrow \exists y(\kappa(y) \wedge \lambda(\text{Dan}, y))$.

Taking this with our previous result above, we can apply *modus ponens* and obtain the expression $\exists y(\kappa(y) \wedge \lambda(\text{Dan}, y))$.

Since we have produced $\exists y(\kappa(y) \wedge \lambda(\text{Dan}, y))$ and "Dan" is a specific object, we can use *existential introduction* to conclude $\exists x\exists y(\kappa(y) \wedge \lambda(x, y))$. QED

2. Fix a universe of discourse containing real and fictional people throughout human history and consider the following argument.

- Only grand wizards have apprentices.
- Not every wizard is grand.
- Therefore, there is a wizard without an apprentice.

Define the following predicates.

$$\xi(x) := \text{"}x \text{ is a wizard.}"$$

$$\zeta(x) := \text{"}x \text{ is grand.}"$$

$$\alpha(x, y) := \text{"}x \text{ is an apprentice of } y\text{"}$$

We now formalize the argument below.

- $\forall x(\exists y(\alpha(y, x)) \rightarrow (\xi(x) \wedge \zeta(x)))$.
- $\neg\forall x(\xi(x) \rightarrow \zeta(x))$.
- Therefore, $\exists x(\xi(x) \wedge \neg\exists y(\alpha(y, x)))$.

We will now prove the third sentence from the first two.

Proof. Assume $\forall x(\exists y(\alpha(y, x)) \rightarrow (\xi(x) \wedge \zeta(x)))$, and assume $\neg\forall x(\xi(x) \rightarrow \zeta(x))$.

Notice that $\neg\forall x(\xi(x) \rightarrow \zeta(x)) \equiv \exists x(\neg(\xi(x) \rightarrow \zeta(x)))$ by the *negation of quantifiers* theorem. Applying *existential elimination*, we then know $\neg(\xi(t) \rightarrow \zeta(t))$ for some person t . We can now observe the following deduction.

$$\begin{aligned} \neg(\xi(t) \rightarrow \zeta(t)) &\equiv \neg(\neg\xi(t) \vee \zeta(t)) && \text{by conditional disintegration} \\ &\equiv \neg\neg\xi(t) \wedge \neg\zeta(t) && \text{by De Morgan's laws} \\ &\equiv \xi(t) \wedge \neg\zeta(t) && \text{by double negation} \end{aligned}$$

From this, we obtain both $\xi(t)$ and $\neg\zeta(t)$ by *conjunction elimination*. We can now say $\neg\xi(t) \vee \neg\zeta(t)$ by *disjunction introduction*.

Recalling our first assumption, we can apply *universal elimination* to derive $\exists y(\alpha(y, t)) \rightarrow (\xi(t) \wedge \zeta(t))$. The time has now come for another observation.

$$\begin{aligned} \exists y(\alpha(y, t)) \rightarrow (\xi(t) \wedge \zeta(t)) &\equiv \neg(\xi(t) \wedge \zeta(t)) \rightarrow \neg\exists y(\alpha(y, t)) && \text{by 1.b. in PSO1} \\ &\equiv (\neg\xi(t) \vee \neg\zeta(t)) \rightarrow \neg\exists y(\alpha(y, t)) && \text{by De Morgan's laws} \end{aligned}$$

Now that we have $\neg\xi(t) \vee \neg\zeta(t)$ and $(\neg\xi(t) \vee \neg\zeta(t)) \rightarrow \neg\exists y(\alpha(y, t))$, we can deduce $\neg\exists y(\alpha(y, t))$ by *modus ponens*. Recalling $\xi(t)$, we then obtain $\xi(t) \wedge \neg\exists y(\alpha(y, t))$ by *conjunction introduction*.

Since t was a particular object² from our universe of discourse, we can conclude $\exists x(\xi(x) \wedge \neg\exists y(\alpha(y, x)))$ by *existential introduction* as desired. QED

²Recall that t was introduced using *existential elimination*, so this is (one of) "the" term(s) that satisfies the statement $\neg(\xi(t) \rightarrow \zeta(t))$.

3. Fix a universe of discourse containing the ancient Greek scholars and consider this.

- Every man is mortal.
- Socrates is a man.
- Therefore, someone is mortal.

Define the following predicates.

$$\mu(x) := "x \text{ is a man.}"$$

$$\omega(x) := "x \text{ is mortal.}"$$

We now formalize the argument below.

- $\forall x(\mu(x) \rightarrow \omega(x))$.
- $\mu(\text{Socrates})$.
- Therefore, $\exists x(\omega(x))$.

We will now prove the third sentence from the first two.

Proof. Assume $\forall x(\mu(x) \rightarrow \omega(x))$ and also $\mu(\text{Socrates})$. By *universal elimination*, we then know $\mu(\text{Socrates}) \rightarrow \omega(\text{Socrates})$. We then have $\omega(\text{Socrates})$ by *modus ponens*. Therefore, applying *existential introduction*, we conclude $\exists x(\omega(x))$. QED