Exoterica 0

Discrete Structures

2^{nd} day of February of the year of our Lord 2026

1. Let φ be a unary predicate.

   a. We will show that $\exists x(\neg\varphi(x)) \vdash \neg\forall x(\varphi(x))$.

   Proof. Assume $\exists x(\neg\varphi(x))$. We then know that $\neg\varphi(t)$ for some t by *existential elimination.* Towards a contradiction, assume $\neg\neg\forall x(\varphi(x))$, which is equivalent to $\forall x(\varphi(x))$ by the *double negation* theorem. Then, by *universal elimination,* we know $\varphi(t)$. However, this contradicts our earlier result $\neg\varphi(t)$. ↯

   Therefore, we have $\neg\forall x(\varphi(x))$ by *reductio ad absurdum.* QED

   b. We will show that $\forall x(\neg\varphi(x)) \vdash \neg\exists x(\varphi(x))$.

   Proof. Assume $\forall x(\neg\varphi(x))$. Towards a contradiction, assume $\neg\neg\exists x(\varphi(x))$, which is equivalent to $\exists x(\varphi(x))$ by the *double negation* theorem. We can then say $\varphi(t)$ for some t by *existential elimination.* Now, applying *universal elimination* to our original assumption yields $\neg\varphi(t)$, contradicting $\varphi(t)$. ↯

   Therefore, we can conclude $\neg\exists x(\varphi(x))$ by *reductio ad absurdum.* QED

2. Let φ be a unary predicate.

   a. We will show that $\neg\exists x(\varphi(x)) \vdash \forall x(\neg\varphi(x))$.

   Proof. Assume $\neg\exists x(\varphi(x))$. Let t be a term. Towards a contradiction, assume $\neg\neg\varphi(t)$, which is equivalent to $\varphi(t)$ by the *double negation* theorem. By *existential introduction,* we then know $\exists x(\varphi(x))$. However, this contradicts our assumption that $\neg\exists x(\varphi(x))$. ↯ Thus, by *reductio ad absurdum,* we have $\neg\varphi(t)$.

   Therefore, we can conclude $\forall x(\neg\varphi(x))$ by *universal introduction.*[1] QED

   [1]This is because we proved $\neg\varphi(t)$, and we introduced t as an ***arbitrary*** *term.*

   b. We will show that $\neg\forall x(\varphi(x)) \vdash \exists x(\neg\varphi(x))$.

   Proof. Assume $\neg\forall x(\varphi(x))$. Towards a contradiction, assume $\neg\exists x(\neg\varphi(x))$. By the previous problem, this implies $\forall x(\neg\varphi(x))$, clearly contradicting the assumption we just made. ↯

   Therefore, we conclude that $\exists x(\neg\varphi(x))$ as desired. QED

3. Let φ be a unary predicate and let ψ be a proposition.

   a. We will show that $\forall x(\varphi(x) \to \psi) \vdash \exists x(\varphi(x)) \to \psi$.

   Proof. Assume $\forall x(\varphi(x) \to \psi)$. Towards a contradiction, assume that we know $\neg\Big(\exists x(\varphi(x)) \to \psi\Big)$. Observe the following deduction.

$$
\begin{aligned}
\neg\Big(\exists x(\varphi(x)) \to \psi\Big) &\equiv \neg\Big(\neg\exists x(\varphi(x)) \lor \psi\Big) && \text{by \textit{conditional disintegration}} \\
&\equiv \neg\neg\exists x(\varphi(x)) \land \neg\psi && \text{by \textit{De Morgan's laws}} \\
&\equiv \exists x(\varphi(x)) \land \neg\psi && \text{by \textit{double negation}}
\end{aligned}
$$

From this, we can derive $\exists x\big(\varphi(x)\big)$ and $\neg\psi$ by *conjunction elimination.* By *existential elimination,* we can now assert $\varphi(t)$ for some t. Recall that $\forall x\big(\varphi(x) \to \psi\big)$. By applying *universal elimination,* we obtain $\varphi(t) \to \psi$. We then know ψ by *modus ponens,* contradicting our prior result $\neg\psi$. ↯

Therefore, we conclude $\exists x\big(\varphi(x)\big) \to \psi$ by *reductio ad absurdum.* QED

b. We will show that $\exists x\big(\varphi(x)\big) \to \psi \vdash \forall x\big(\varphi(x) \to \psi\big)$.

Proof. Assume $\exists x\big(\varphi(x)\big) \to \psi$. We know this is equivalent to $\neg\exists x\big(\varphi(x)\big) \vee \psi$ by *conditional disintegration.* We now have two cases.[2]

[2] We are setting up a use of the *disjunction elimination* theorem from `PS02`. This theorem is also known as ***proof by cases***.

Case 1:
Suppose that $\neg\exists x\big(\varphi(x)\big)$. Then, by problem 2.a., we know $\forall x\big(\neg\varphi(x)\big)$. Towards a contradiction, assume $\neg\forall x\big(\varphi(x) \to \psi\big)$. By problem 2.b., we then get $\exists x\Big(\neg\big(\varphi(x) \to \psi\big)\Big)$. We can now observe the following.

$$\begin{aligned} \exists x\Big(\neg\big(\varphi(x) \to \psi\big)\Big) &\equiv \exists x\Big(\neg\big(\neg\varphi(x) \vee \psi\big)\Big) && \text{by } \textit{conditional disintegration} \\ &\equiv \exists x\big(\neg\neg\varphi(x) \wedge \neg\psi\big) && \text{by } \textit{De Morgan's laws} \\ &\equiv \exists x\big(\varphi(x) \wedge \neg\psi\big) && \text{by } \textit{double negation} \end{aligned}$$

We then know $\varphi(t) \wedge \neg\psi$ by *existential elimination,* which implies $\varphi(t)$ by *conjunction elimination.* Now, recalling that $\forall x\big(\neg\varphi(x)\big)$, we know $\neg\varphi(t)$ by *universal elimination.* ↯

Therefore, $\forall x\big(\varphi(x) \to \psi\big)$ by *reductio ad absurdum.*

Case 2:
Suppose ψ. Let t be an arbitrary term. We can then say $\varphi \vee \neg\varphi(t)$ by *disjunction introduction,* which is equivalent to $\neg\varphi(t) \vee \psi$ by *commutativity.* This is then equivalent to $\varphi(t) \to \psi$ by *conditional disintegration.* Thus, $\forall x\big(\varphi(x) \to \psi\big)$ by *universal introduction.*

Since we were able to show that $\neg\exists x\big(\varphi(x)\big) \vdash \forall x\big(\varphi(x) \to \psi\big)$, and we also showed that $\psi \vdash \forall x\big(\varphi(x) \to \psi\big)$, and since we know $\neg\exists x\big(\varphi(x)\big) \vee \psi$, we can apply *disjunction elimination* and conclude $\forall x\big(\varphi(x) \to \psi\big)$ as desired. QED