12 Graph Theory

Definition 12.1: Simple Graphs.

$V(G)$ $E(G)$ A *simple graph* G is a connectivity structure[1] determined by a *set of vertices* $V(G)$ and a *set of edges* $E(G)$ satisfying the following two properties.

1. $|V(G)| < |\mathbb{N}|$[2]
2. $E(G) \subseteq \{p \mid p \subseteq V(G) \wedge |p| = |2|\}$

The first condition establishes that simple graphs have finitely many nodes. The second condition describes what edges are like: the elements of $E(G)$ must be subsets of $V(G)$ with cardinality 2. There are several implications of this condition.

1. Edges are *undirected* because they are *unordered pairs* $\{u, v\}$, where $u, v \in V(G)$.
2. There are no *self-edges* $\{v, v\}$ for any $v \in V(G)$ because $|\{v, v\}| \neq |2|$.
3. There can be *at most one* edge between any two nodes.

graph When we say *graph* without qualification, we mean *simple graph* as defined above. ⌟

As an example, consider the graph below that has one node selected as an exemplar.

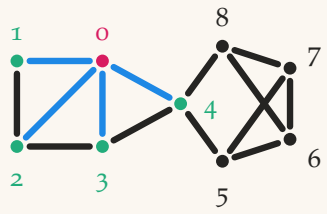

Figure 1. An example of a graph G with $V(G) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and edge set $E(G)$ as shown below.

$\{0,1\}$	$\{0,2\}$	$\{1,2\}$	$\{0,3\}$
$\{0,4\}$	$\{3,4\}$	$\{4,5\}$	$\{4,8\}$
$\{5,6\}$	$\{5,7\}$	$\{6,7\}$	$\{6,8\}$

The node 0 is highlighted in *red*, and its neighbors in $N_G(0)$ are highlighted in *green*. The edges on its coboundary $\delta_G(0)$ are colored in *blue*.

$N_G(v)$ $\delta_G(v)$ Given a graph G and a node $v \in V(G)$, as in the example above, we define the *neighborhood* $N_G(v)$ *of* v *in* G to be the set $\{w \in V(G) \mid \{v, w\} \in E(G)\}$. This is the set of all nodes w that are connected to v through an edge $\{v, w\}$ in $E(G)$. We similarly define the *coboundary* $\delta_G(v)$ *of* v *in* G to be the set $\{e \in E(G) \mid v \in e\}$ of all edges that involve the node v.[3]

Associated with every graph G is a function $\mathsf{deg}_G : V(G) \to \mathbb{N}$ that computes for each node $v \in G$ the number of edges from G incident on v as shown below.

$\mathsf{deg}_G(v)$
$$\mathsf{deg}_G(v) := \#\delta_G(v)$$

Because the graphs we are considering do not have *"multiedges,"* it turns out that the number of neighbors a vertex has is equal to its degree.

Lemma 12.1.
For any graph G and any $v \in V(G)$, we have $\mathsf{deg}_G(v) = \#N_G(v)$.

Proof. Let G be a graph, and let $v \in V(G)$. Define the function $f : N_G(v) \to \delta_G(v)$ by $f(w) := \{v, w\}$ for each $w \in N_G(v)$. We will now quickly prove f is a bijection.

First, let $u, w \in N_G(v)$ and suppose $f(u) = f(w)$. Now observe.

$$\begin{aligned} f(u) = f(w) &\Rightarrow \{u, v\} = \{w, v\} \\ &\Rightarrow \{u, v\} \setminus \{v\} = \{w, v\} \setminus \{v\} \\ &\Rightarrow \{u\} = \{w\} \\ &\Rightarrow u = v \end{aligned}$$

Thus, we have that f is injective.

[1] Of course, everything must be a set in ZFC, so a graph is actually a set. This is typically encoded by $G = (V, E)$, where V and E are the vertex and edge sets of G respectively.

[2] People *typically* insist on the additional requirement that $V(G) \neq \varnothing$, though this is not *always* implied. In order to disambiguate, we should be sure to keep the empty set in mind as a possibility when discussing any statement that universally quantifies over *"(simple) graphs."*

[3] When a node v participates in an edge e, which formally means $v \in e$ with our definitions, we often say that the node v is *"incident on"* the edge e.

Second, let $e \in \delta_G(v)$ and note that $v \in e$ by definition. Now, since $\delta_G(v) \subseteq E(G)$, we can see $e \in E(G)$, which means $e \subseteq V(G)$ and $|e| = |2|$. This means there exists some $x \in V(G)$ such that $\{x, v\} = e$, which immediately implies $x \in N_G(v)$. We can now simply observe that $f(x) = \{x, v\} = e$ as desired. Thus, f is surjective.

Since f is both injective and surjective, f is a bijection from $N_G(v)$ to $\delta_G(v)$, and we therefore have $|N_G(v)| = |\delta_G(v)|$, letting us conclude $\texttt{deg}_G(v) = \#\delta_G(v) = \#N_G(v)$.

QED

Theorem 12.1: The Handshaking Problem.
In any graph G, there exist $u, v \in V(G)$ such that $u \neq v$ but $\texttt{deg}_G(u) = \texttt{deg}_G(v)$.

Proof. // see lecture[1]

[1] The proof will be written here soon.

QED

Lemma 12.2: The Handshake Lemma.
For any graph G, we have $\displaystyle\sum_{v \in V(G)} \texttt{deg}_G(v) = 2 \cdot \#E(G)$. ⌟

12.1 *Preservation of Structure and Substructure*

Definition 12.2: Homomorphisms of Graphs.

homomorph.

Given graphs G and H, a *graph homomorphism* from G to H is a mapping of the nodes of G to nodes in H in such a way that the *adjacency structure of G is preserved within H*. Formally, this is a function $f : V(G) \to V(H)$ with the following property.

$$\big(\forall u, v \in V(G)\big)\Big(\{u, v\} \in E(G) \;\Rightarrow\; \{f(u), f(v)\} \in E(H)\Big)$$

The typical practice is to write $f : G \to H$ when f is a homomorphism between two graphs G and H, with the understanding that the domain and codomain of f are technically really $V(G)$ and $V(H)$ respectively.

monomorph. *monic*

A *monomorphism of graphs* is a *left-cancellative* graph homomorphism. This means that we call f *monic iff,* for any graph X and any pair of homomorphisms $g_1 : X \to G$ and $g_2 : X \to G$, if $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$. In particular, we should think that ***graph monomorphisms are injective graph homomorphisms***. When we know f is a monomorphism from G to H, we often abbreviate this by writing $f : G \hookrightarrow H$.

$f : G \hookrightarrow H$

epimorph. *epic*

An *epimorphism of graphs* is a *right-cancellative* graph homomorphism. This means that we call f *epic iff,* for any graph Y and any pair of homomorphisms $g_1 : H \to Y$ and $g_2 : H \to Y$, if $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$. In particular, this implies that ***graph epimorphisms are surjective graph homomorphisms***. When f is understood to be an epic morphism, you may encounter the notation $f : G \twoheadrightarrow H$.

$f : G \twoheadrightarrow H$

isomorph.

Finally, an *isomorphism of graphs* is a *fully invertible* graph homomorphism. More precisely, we call a homomorphism $f : G \to H$ an *isomorphism* when there exists a homomorphism $g : H \to G$ such that $g \circ f = \texttt{id}_G$ and $f \circ g = \texttt{id}_H$. In such a situation, the notation $f^{-1} \coloneqq g$ denotes *the (unique) two-sided inverse* of f, which will also be an isomorphism. We can thus conceptualize that ***graph isomorphisms are bijective graph homomorphisms***.

When there is an isomorphism between two graphs G and H, we say that G is *isomorphic* to H and denote this by $G \cong H$. ⌟

$G \cong H$

12.2 Distinguished Heraldry

Definition 12.3: Induced Subgraphs.

subgraph Let G be a graph. We say that a graph H is a *subgraph of* G, which we denote $H \subseteq G$, precisely when $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

$G[W]$ Given a set of vertices $W \subseteq V(G)$, the *subgraph induced by* W is the graph $G[W]$ determined by $V(G[W]) := W$ and $E(G[W]) = \{\{u, v\} \in E(G) \mid u \in W \wedge v \in W\}$. ⌟

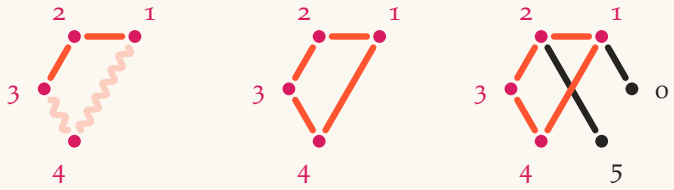

Figure 2. An example of a graph G on the right with two of its subgraphs shown in the middle and on the left. The subset $W := \{1, 2, 3, 4\} \subseteq V(G)$ is highlighted in *red*. The edges involved in the *subgraph induced by* W are highlighted in *orange*. The left-most subgraph only contains the edges $\{1, 2\}$ and $\{2, 3\}$, but is missing $\{3, 4\}$ and $\{1, 4\}$, so it *is not* an induced subgraph of G, while the subgraph in the middle *is* the induced subgraph $G[W]$.

Lemma 12.3.
For any graph G and subgraph $H \subseteq G$, we have $(\forall v \in V(H))(\deg_H(v) \leqslant \deg_G(v))$.

Proof. Let G and H be graphs such that $H \subseteq G$, and let $v \in V(H)$. By recalling how the degree of a node is defined, we can see the following two statements are *true*.

$$\deg_H(v) = \#\delta_H(v) = \#\Big\{e \in E(H) \;\Big|\; v \in e\Big\}$$
$$\deg_G(v) = \#\delta_G(v) = \#\Big\{e \in E(G) \;\Big|\; v \in e\Big\}$$

Since $H \subseteq G$, we know $E(H) \subseteq E(G)$ by definition. This implies $\delta_H(v) \subseteq \delta_G(v)$, and so $|\delta_H(v)| \leqslant |\delta_G(v)|$. We therefore conclude that $\deg_H(v) \leqslant \deg_G(v)$.

QED

Lemma 12.4.
If G and H are graphs and $f : G \hookrightarrow H$ is a *graph monomorphism* (*i.e.,* an injective graph homomorphism), then there exists a subgraph $J \subseteq H$ such that $G \cong J$.

Definition 12.4: Complementary Graphs.
Let G be a graph, and recall that the set of all *possible* edges in G is given by the set $\{p \mid p \subseteq V(G) \wedge |p| = |2|\}$. This is the set of all unordered pairs $\{u, v\}$ such that $u \in$
non-edge $V(G)$ and $v \in V(G)$ and $u \neq v$. Such an unordered pair $\{u, v\}$ is a *non-edge* in G when
$\overline{E(G)}$ $\{u, v\} \notin E(G)$. The set of all non-edges in G—denoted by $\overline{E(G)}$—is defined to be $\{p \mid p \subseteq \mathbb{P}(V(G)) \wedge |p| = |2|\} \setminus E(G)$. This is sometimes referred to as *"the (relative) complement of $E(G)$ (within G)."*

$\overline{G}$ Given a graph G, we define the *complement of* G to be the graph $\overline{G}$ given by the vertex set $V\left(\overline{G}\right) := V(G)$ and the edge set $E\left(\overline{G}\right) := \overline{E(G)}$. ⌟

Definition 12.5: Path Graphs.
P_k Given a natural number $k \in \mathbb{N}$, the *path graph* P_k *of length* k is determined by the vertices $V(P_k) := \{i \in \mathbb{N} \mid i \leqslant k\}$ and the edges $E(P_k) := \{\{i, i+1\} \mid i \in \mathbb{N} \wedge i < k\}$.

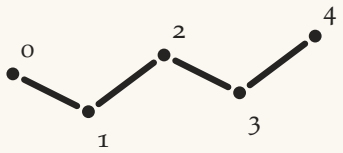

Figure 3. A visual depiction of P_4, the path graph of length 4. Notice that the length of a path is measured by counting *edges*, not nodes.

⌟

Definition 12.6: Walks and Paths.

walk Let G be a graph, let $k \in \mathbb{N}$, and consider $u, v \in V(G)$. A *walk of length k* in G from u to v is a graph homomorphism $w : P_k \to G$ such that $w(0) = u$ and $w(k) = v$.

path A *path of length k* in G from u to v is a *monomorphism* $p : P_k \hookrightarrow G$ such that $p(0) = u$ and $p(k) = v$. In other words, a path from u to v is simply an *injective walk* from u to v, meaning it is a walk that starts at u, ends at v, and does not repeat vertices. ⌟

Definition 12.7: Cycle Graphs.

C_k Given $k \in \mathbb{N}$ such that $k \geqslant 3$, the *cycle graph C_k of length k* is the graph given by $V(C_k) \coloneqq \{i \in \mathbb{N} \mid i < k\}$ and $E(C_k) \coloneqq \{\{i, i+1\} \mid i \in \mathbb{N} \wedge i < k-1\} \cup \{k-1, 0\}$.

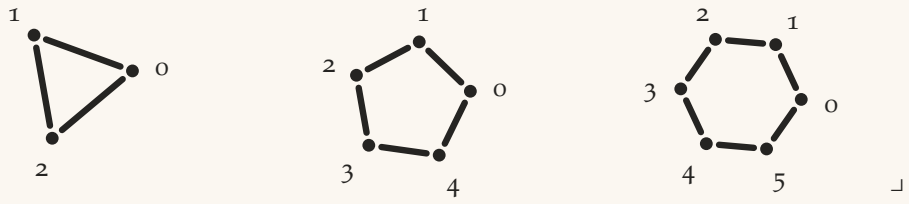

Figure 4. Visualizations of the cycles C_3, C_5, and C_6 of lengths 3, 5, and 6. ⌟

Definition 12.8: Circuits and Cycles.

circuit Let G be a graph, and let $k \in \mathbb{N}$ such that $k \geqslant 3$. A *circuit of length k* in G is a graph homomorphism $t : C_k \to G$.[1]

cycle A *cycle of length k* in G is a *monomorphism* $c : C_k \hookrightarrow G$. This is analogous to how we defined paths based on walks; a cycle is an *injective* circuit. ⌟

[1] *N.B.* This may not be consistent with the way a *"circuit"* is defined in most other contexts—there is often an additional requirement that *edges* are not repeated in a *"circuit."* We will not place much emphasis on this distinction. Circuits are only really mentioned here as a passing curiosity on the way to defining *cycles,* which are a fundamental concept in graph theory.

Definition 12.9: Complete Graphs.

K_n Given $n \in \mathbb{N}$, the *complete graph K_n* has the node set $V(K_n) \coloneqq \{i \in \mathbb{N} \mid i < n\}$ and has all possible edges $E(K_n) \coloneqq \{\{i, j\} \mid i \in V(K_n) \wedge j \in V(K_n) \wedge i \neq j\}$ between them.

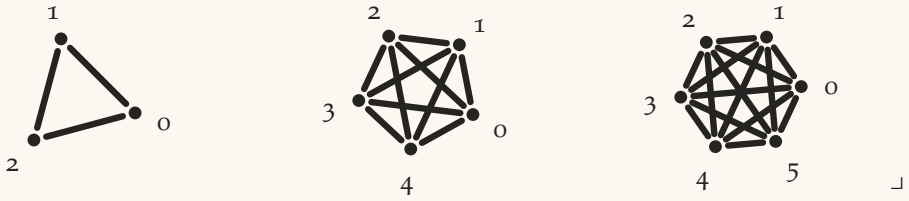

Figure 5. An illustration of the complete graphs K_3, K_5, and K_6 on 3, 5, and 6 nodes respectively. ⌟

Definition 12.10: Cliques.

clique A *clique of size n* in G, where $n \in \mathbb{N}$, is a subgraph of G picked out by a graph homomorphism $k : K_n \to G$. ⌟

This definition above of *"cliques"* for graphs is appropriate for the following reason. For any $n \in \mathbb{N} \setminus \{0, 1\}$, if $f : K_n \to G$ is a graph homomorphism from the complete graph K_n to an arbitrary graph G, then f *must* be injective, and G therefore contains a subgraph isomorphic to K_n. This follows thanks to the lemma below.

Lemma 12.5.

Let G and H be graphs, and let $f : G \to H$ be an arbitrary graph homomorphism. Then, $(\forall u, v \in V(G))(\{u, v\} \in E(G) \Rightarrow f(u) \neq f(v))$. ⌟

12.3 Connectivity

Definition 12.11: Connectivity.

connected

We say that a graph G is *connected iff* for every pair of nodes $u, v \in V(G)$, there exists a path in G from u to v. Formally, this means there exists $k \in \mathbb{N}$ and a homomorphism $p : P_k \to G$ such that p is *injective* and satisfies $p(0) = u$ and $p(k) = v$. ⌟

Definition 12.12: Connected Component.

connected component

Given a graph G, a *connected component* of G is a *maximally connected subgraph of G.* More precisely, we say H *is a connected component of* G if and only if H is connected, $H \subseteq G$ and, for any graph J, if $H \subseteq J$ and $J \subseteq G$, then either $H = J$ or $J = G$ or J is disconnected. ⌟

Theorem 12.2: Graphs Are Partitioned by Their Connected Components.

Let G be a graph with $V(G) \neq \varnothing$. Let $C \coloneqq \{H \mid H \text{ is a connected component of } G\}$. Then, the following three conditions hold.

1. $\varnothing \notin C$.
2. For every $H_1, H_2 \in C$, we have $H_1 \cap H_2 = \varnothing$.
3. $\cup C = V(G)$; in other words, this means $\bigcup_{H \in C} H = V(G)$.

Further, $|C| \leqslant |V(G)|$. This implies in particular that there are only finitely many connected components in G. ⌟

Theorem 12.3.

Every graph G contains a path of maximal length.[1]

[1] This is sometimes phrased as *"every simple graph contains a longest path."*

Proof. henlo

QED