

3 The Rules of Inference

Given a natural number n and $n + 1$ propositions $p_0, p_1, p_2, \dots, p_n$, the syntax of a typical rule of inference generally follows the schema shown below.¹

$$p_0, p_1, \dots, p_{n-1} \vdash p_n$$

$\vdash$ Crucially, the rules of inference invoke the use of a new meta-symbol $\vdash$ separating a (possibly empty) list of propositions on the left from a proposition on the right. This is meant to denote that, by *assuming* all of the propositions $p_0, p_1, \dots, p_{n-1}$ on the left, we can *prove* the proposition p_n on the right follows by relying only on:

1. any axioms we have assumed,
2. the rules of inference,
3. any other theorems we have already proven.

This symbol $\vdash$ is often called a “turnstile” because of its shape, but we often read it as “proves” due to its role in denoting *syntactic entailment*,² also known as *provability*.

$\dashv\vdash$ When we have two propositions p and q such that $p \vdash q$ and $q \vdash p$, we can denote this in a more compact way by writing $p \dashv\vdash q$.

¹ We will adopt the convention that any axioms we specify in our logical system are implicitly assumed on the left-hand side of the $\vdash$ symbol, even if we don’t write them.

² Note that this is a distinct concept from *semantic entailment*, which is usually denoted by the $\models$ symbol and speaks about *satisfiability* in terms of *truth values*.

3.1 Rules for Zeroth-Order Expressions

To make these rules easier to read at a glance, we will simplify our notation for the presentation of these rules of inference by adopting the following convention within this section: whenever a rule of inference in this section uses the symbols p , q , or r , we will understand those symbols to refer to *arbitrary propositions*.

Implication Rules

Rule 1: Implication Introduction.

$$(p \vdash q) \vdash p \rightarrow q \quad \lrcorner$$

Rule 2: Implication Elimination.

$$p, p \rightarrow q \vdash q \quad \lrcorner$$

Equivalence Rules

Rule 3: Biconditional Introduction.

$$p, q \vdash p \leftrightarrow q \quad \lrcorner$$

Rule 4: Biconditional Elimination.

$$p \leftrightarrow q \vdash (p \rightarrow q) \wedge (q \rightarrow p) \quad \lrcorner$$

Conjunction Rules

Rule 5: Conjunction Introduction.

$$p, q \vdash p \wedge q$$

┌

Rule 6: Conjunction Elimination.

$$p \wedge q \vdash p$$

$$p \wedge q \vdash q$$

┌

Disjunction Rules

Rule 7: Disjunction Introduction.

$$p \vdash p \vee q$$

$$q \vdash p \vee q$$

┌

Rule 8: Disjunction Elimination.

$$(p \vee q), (p \rightarrow r), (q \rightarrow r) \vdash r$$

┌

Negation Rules

Rule 9: Negation Introduction.

$$(p \rightarrow q), (p \rightarrow \neg q) \vdash \neg p$$

┌

Rule 10: Negation Elimination.

$$\neg \neg p \vdash p$$

┌

Truth Rules

Rule 11: Truth Introduction.

$$\vdash \top$$

┌

Rule 12: Law of the Excluded Middle.

$$\vdash \neg p \vee p$$

┌

Falsity Rules

Rule 13: False Introduction.

$$\neg p \wedge p \vdash \perp$$

┌

Rule 14: False Elimination.

$$\perp \vdash p$$

┌

3.2 Theorems of the Propositional Calculus

Theorem 3.1.

For any propositions p and q , we have the following deduction.

$$p \rightarrow q \vdash (p \vdash q)$$

┌

Theorem 3.2: Conditional Disintegration.

For any propositions p and q , we have the following equivalence.

$$p \rightarrow q \dashv\vdash \neg p \vee q$$

┌

Theorem 3.3: Contrapositive Equivalence.

For any propositions p and q , we have the following equivalence.

$$p \rightarrow q \dashv\vdash \neg q \rightarrow \neg p \quad \lrcorner$$

Theorem 3.4: Exportation.

For any propositions p , q , and r , we have the following equivalence.

$$(p \wedge q) \rightarrow r \dashv\vdash p \rightarrow (q \rightarrow r) \quad \lrcorner$$

Theorem 3.5: Absorption.

For any propositions p and q , we have the following equivalence.

$$p \rightarrow q \dashv\vdash p \rightarrow (p \rightarrow q) \quad \lrcorner$$

Theorem 3.6: Commutativity of Conjunctions and Disjunctions.

For any propositions p and q , the following equivalences hold.

$$\begin{aligned} p \wedge q &\dashv\vdash q \wedge p \\ p \vee q &\dashv\vdash q \vee p \end{aligned} \quad \lrcorner$$

Theorem 3.7: Associativity of Conjunctions and Disjunctions.

For any propositions p , q , and r , the following equivalences hold.

$$\begin{aligned} p \wedge (q \wedge r) &\dashv\vdash (p \wedge q) \wedge r \\ p \vee (q \vee r) &\dashv\vdash (p \vee q) \vee r \end{aligned} \quad \lrcorner$$

Theorem 3.8: Distributivity of Conjunctions and Disjunctions.

For any propositions p , q , and r , the following equivalences hold.

$$\begin{aligned} p \wedge (q \vee r) &\dashv\vdash (p \wedge q) \vee (p \wedge r) \\ p \vee (q \wedge r) &\dashv\vdash (p \vee q) \wedge (p \vee r) \end{aligned} \quad \lrcorner$$

Theorem 3.9: Destructive Dilemma.

For any propositions p , q , r , and s the following equivalence holds.

$$(p \rightarrow r), (q \rightarrow s), (\neg r \vee \neg s) \vdash (\neg p \vee \neg q) \quad \lrcorner$$

Theorem 3.10: De Morgan's Theorems.

For any propositions p and q , the following two equivalences hold.

$$\begin{aligned} \neg(p \wedge q) &\dashv\vdash \neg p \vee \neg q \\ \neg(p \vee q) &\dashv\vdash \neg p \wedge \neg q \end{aligned} \quad \lrcorner$$

3.3 Rules for First-Order Expressions

In this section, as in the previous section, we will adopt the convention that φ stands for an arbitrary well-formed formula in order to make the rules easier to read.

Universal Rules

Rule 15: Universal Introduction.

$\varphi(t)$ for an arbitrary term $t \vdash \forall x(\varphi(x))$ ┘

Rule 16: Universal Elimination.

$\forall x(\varphi(x)) \vdash \varphi(t)$ for any term t ┘

Existential Rules

Rule 17: Existential Introduction.

$\varphi(t)$ for some term $t \vdash \exists x(\varphi(x))$ ┘

Rule 18: Existential Elimination.

$\exists x(\varphi(x)) \vdash \varphi(t)$ for some new term t that has not yet appeared in this context ┘

3.4 Theorems of the Predicate Calculus

Theorem 3.11: Quantifier Shift.

The following equivalences regarding repeated quantifiers hold for any wff φ .

$$\begin{aligned}\forall x \forall y (\varphi(x, y)) &\dashv\vdash \forall y \forall x (\varphi(x, y)) \\ \exists x \exists y (\varphi(x, y)) &\dashv\vdash \exists y \exists x (\varphi(x, y))\end{aligned}$$

We also have the following for any existentially quantified universal wff φ .

$$\exists x \forall y (\varphi(x, y)) \vdash \forall y \exists x (\varphi(x, y))$$

However, we *cannot* generally infer an existentially quantified universal from a universally quantified existential. The invalidity of this kind of reasoning is stated below for any wff φ .

$$\forall x \exists y (\varphi(x, y)) \not\vdash \exists y \forall x (\varphi(x, y)) \quad \text{┘}$$

Theorem 3.12: Negation of Quantifiers.

The following equivalences hold for any wff φ .

$$\begin{aligned}\neg \forall x (\varphi(x)) &\dashv\vdash \exists x (\neg \varphi(x)) \\ \neg \exists x (\varphi(x)) &\dashv\vdash \forall x (\neg \varphi(x))\end{aligned} \quad \text{┘}$$

Theorem 3.13: Distribution of Quantifiers.

Let φ be an arbitrary wff, and let p be an arbitrary proposition that does not involve the symbol “ x ”. The following equivalences then hold — be mindful of parentheses.

$$\begin{aligned}\forall x (\varphi(x) \wedge p) &\dashv\vdash \forall x (\varphi(x)) \wedge p \\ \exists x (\varphi(x) \wedge p) &\dashv\vdash \exists x (\varphi(x)) \wedge p\end{aligned}$$

We also have similar equivalences for disjunctive statements.

$$\forall x(\varphi(x) \vee p) \dashv\vdash \forall x(\varphi(x)) \vee p$$

$$\exists x(\varphi(x) \vee p) \dashv\vdash \exists x(\varphi(x)) \vee p$$

Conditional statements involving a quantifier in the *consequent* behave similarly.

$$\forall x(p \rightarrow \varphi(x)) \dashv\vdash p \rightarrow \forall x(\varphi(x))$$

$$\exists x(p \rightarrow \varphi(x)) \dashv\vdash p \rightarrow \exists x(\varphi(x))$$

Conditional statements with a quantified *antecedent*, however, are more nuanced. Additional parentheses have been added for clarity.

$$\forall x(\varphi(x) \rightarrow p) \dashv\vdash (\exists x(\varphi(x))) \rightarrow p$$

$$\exists x(\varphi(x) \rightarrow p) \dashv\vdash (\forall x(\varphi(x))) \rightarrow p$$

⌋