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Theorem: For any graph $\mathcal{G}$, we have $2\cdot|E(\mathcal{G})| = \sum_{v\in V(\mathcal{G})} \deg_{\mathcal{G}}(v)$.

Proof: We will prove that $(\forall n\in\mathbb{N})\Big(\forall G\big((G \text{ is a graph} \wedge |V(G)|=n) \Rightarrow 2\cdot|E(G)| = \sum_{v\in V(G)}\deg_G(v)\big)\Big)$ by weak induction (essentially, induction on the order of G).

<u>Basis Step</u>: Let G be a graph and assume $|V(G)|=0$, so that $V(G)=\varnothing$. Notice that $\forall x(x\subseteq\varnothing \Rightarrow x=\varnothing \Rightarrow |x|=0)$, so $E(G)\subseteq\{w \mid w\subseteq V(G)\wedge |w|=2\}=\varnothing$, implying that $E(G)=\varnothing$. Therefore, we can see:

$$2\cdot|E(G)| = 2\cdot 0 = 0 = \sum_{v\in\varnothing}\deg_G(v) = \sum_{v\in V(G)}\deg_G(v).$$

<u>Inductive Step</u>: Let $k\in\mathbb{N}$ and assume $\forall G\big((G \text{ is a graph} \wedge |V(G)|=k) \Rightarrow 2\cdot|E(G)| = \sum_{v\in V(G)}\deg_G(v)\big)$.

Now, let G be a graph such that $|V(G)|=k+1$. Since $k+1>k\geq 0$, we know $V(G)\neq\varnothing$, so there exists $x\in V(G)$. Let's see what happens if we remove x from G. Consider the graph H given by

$$\begin{aligned} V(H) &:= V(G)\setminus\{x\} \\ E(H) &:= E(G)\setminus I_G(x) \\ &= \{e\in E(G) \mid x\notin e\}. \end{aligned}$$

First, notice that vertices outside of the neighborhood of x <u>who are not x</u> have the same degree in H as they did in G.

For any $v\in V(G)\setminus(N_G(x)\cup\{x\})$, we can observe:

$$\begin{aligned} \deg_G(v) &= |I_G(v)| \\ &= |\{e\in E(G) \mid v\in e\}| \\ &= |\{e\in E(G) \mid v\in e \wedge x\notin e\}| \quad \text{because } v\notin N_G(x) \text{, which implies } (\forall e\in E(G))(v\in e\Rightarrow x\notin e). \\ &= |\{e\in E(H) \mid v\in e\}| \quad \text{because } x\notin V(H) \text{, so } (\forall e\in E(H))(x\notin e)\text{... it's also literally the definition of } E(H) \text{ we gave above.} \\ &= |I_H(v)| \\ &= \deg_H(v). \end{aligned}$$

Second, we can see that the degree of nodes that were connected to x decreases by precisely the number of edges that connected that node to x.

For any $v\in N_G(x)$, we can observe:

$$\begin{aligned} \deg_G(v) &= |I_G(v)| \\ &= |\{e\in E(G) \mid v\in e\}| \\ &= |\{e\in E(G) \mid v\in e\wedge x\in e\}\cup\{e\in E(G) \mid v\in e\wedge x\notin e\}| \\ &= |\{e\in E(G) \mid v\in e\wedge x\in e\}| + |\{e\in E(G) \mid v\in e\wedge x\notin e\}| \quad \text{Since the sets are disjoint} \\ &= |\{e\in E(G) \mid v\in e\wedge x\in e\}| + |\{e\in E(H) \mid v\in e\}| \\ &= |\{e\in E(G) \mid v\in e\wedge x\in e\}| + |I_H(v)| \\ &= |\{e\in E(G) \mid v\in e\wedge x\in e\}| + \deg_H(v). \end{aligned}$$

[redacted]

Now, notice that $V(G) = N_G(x) \cup (V(G) \setminus (N_G(x) \cup \{x\})) \cup \{x\}$, and these sets are all pairwise disjoint. From this, we can observe the following:

$$\sum_{v \in V(G)} \deg_G(v) = \left(\sum_{v \in N_G(x)} \deg_G(v)\right) + \left(\sum_{v \in V(G) \setminus (N_G(x) \cup \{x\})} \deg_G(v)\right) + \sum_{v \in \{x\}} \deg_G(v)$$

$$= \left(\sum_{v \in N_G(x)} \deg_G(v)\right) + \left(\sum_{v \in V(G) \setminus (N_G(x) \cup \{x\})} \deg_H(v)\right) + \deg_G(x)$$

$$= \left(\sum_{v \in N_G(x)} |\{e \in E(G) \mid v \in e \wedge x \in e\}| + \deg_H(v)\right) + \left(\sum_{v \in V(G) \setminus (N_G(x) \cup \{x\})} \deg_H(v)\right) + \deg_G(x)$$

Split up the sum on the left, and recognize there is exactly one edge between x and any $v \in N_G(x)$.

$$= \left(\sum_{v \in N_G(x)} 1\right) + \left(\sum_{v \in N_G(x)} \deg_H(v)\right) + \left(\sum_{v \in V(G) \setminus (N_G(x) \cup \{x\})} \deg_H(v)\right) + \deg_G(x)$$

$$= |N_G(x)| + \left(\sum_{v \in V(G) \setminus \{x\}} \deg_H(v)\right) + \deg_G(x)$$

$$= |I_G(x)| + \left(\sum_{v \in V(H)} \deg_H(v)\right) + \deg_G(x)$$

$$= \deg_G(x) + \left(\sum_{v \in V(H)} \deg_H(v)\right) + \deg_G(x)$$

$$= 2 \cdot \deg_G(x) + \left(\sum_{v \in V(H)} \deg_H(v)\right)$$

this is by the inductive hypothesis

$$= 2 \cdot |\{e \in E(G) \mid x \in e\}| + 2 \cdot |E(H)|$$

$$= 2 \cdot |\{e \in E(G) \mid x \in e\}| + 2 \cdot |\{e \in E(G) \mid x \notin e\}|$$

$$= 2 \cdot (|\{e \in E(G) \mid x \in e \vee x \notin e\}|)$$

$$= 2 \cdot |E(G)|$$

Therefore, we have $\sum_{v \in V(G)} \deg_G(v) = 2 \cdot |E(G)|$ as desired.

That concludes our inductive step, and so we're done!

QED