

10 Learning How to Count

Definition 10.1: \circ -jections.

Let X and Y be sets and consider a function $f : X \rightarrow Y$. When $x \in X$ and $y \in Y$ such that $f(x) = y$, we usually call y *the image of x (under f)*, and x as *a preimage for y (under f)*.

An injective function always maps distinct inputs to distinct outputs, so that there is *at most one preimage* for any image under f . Formally, we say that f is an *injection* iff $(\forall x_1, x_2 \in X)(x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$. This is equivalent to the contrapositive statement $(\forall x_1, x_2 \in X)(f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$.

A surjective function maps onto its *entire* codomain, so every element of the codomain *has at least one preimage*. Formally, f is a *surjection* iff $(\forall y \in Y)(\exists x \in X)(f(x) = y)$.

A *bijjective* function, as the name implies, is one that is both injective and surjective.

Lemma 10.1.

For any codomain Y , there is a *unique* function $\varphi : \emptyset \rightarrow Y$. This function φ is the empty set \emptyset (sometimes called the “*empty function*”), and φ is necessarily injective.

Lemma 10.2.

For any nonempty domain $X \neq \emptyset$, there are *no* functions $\psi : X \rightarrow \emptyset$.

Definition 10.2: Relative Cardinalities.

Let X and Y be sets. We measure the relative sizes of sets based on the kinds of functions definable between them. We will refer to the “*size*” of a set as its “*cardinality*.”

If it is possible to define an injection between X and Y , then there must be enough room within Y to *spread out* the elements of X without overlapping them. In this situation, we will say that the cardinality of X is no greater than the cardinality of Y . Formally, we define $|X| \leq |Y|$ iff there exists an injective function $f : X \rightarrow Y$.

If it is possible to define a surjection between X and Y , then there must be enough elements in X to cover all of Y without having to send anything to two different places. In this situation, we will say that the cardinality of Y is no greater than the cardinality of X . However, we must make a special exception here when $X \neq \emptyset$ and $Y = \emptyset$; in this situation, there do not exist *any* functions from X to Y , but we would still like to be able to say that the nonempty set X forming the domain is “*larger*” than the empty set Y forming the codomain. Therefore, we will formally define $|X| \geq |Y|$ by cases. When $Y \neq \emptyset$, we will define $|X| \geq |Y|$ to mean that there exists a surjective function $f : X \rightarrow Y$. However, when $Y = \emptyset$, we will simply say that $|X| \geq |\emptyset|$ is always *true* by definition.

Now, when we can find a function between two sets that is *simultaneously* injective and surjective, it should be intuitive by the above reasoning that the sets “*have the same size*.”

Thus, we define $|X| = |Y|$ iff there exists a bijective function $f : X \rightarrow Y$.

We will introduce strict versions of the above relations in the normal expected way.

Define $|X| < |Y| :\Leftrightarrow |X| \leq |Y| \wedge |X| \neq |Y|$, and define $|X| > |Y| :\Leftrightarrow |X| \geq |Y| \wedge |X| \neq |Y|$.

Definition 10.3: Composition.

Let X , Y , and Z be sets. Given two compatible¹ functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, we define the *composition* $g \circ f : X \rightarrow Z$ by $(g \circ f)(x) = g(f(x))$ for all $x \in X$.

¹We say functions are *compatible* when the codomain of one is equal to the domain of the other.

$|X| \leq |Y|$

$|X| \geq |Y|$

$|X| = |Y|$

$|X| < |Y|$

$|X| > |Y|$

Definition 10.4: \circ -morphisms.

Recall that there is a special function $\text{id}_S : S \rightarrow S$ defined on any set S by the mapping $\text{id}_{S(s)} := s$ for all $s \in S$. Keep this in mind; let X and Y be sets, and consider $f : X \rightarrow Y$.

We call f a *monomorphism* when, for any set Z and any pair of functions $g_1 : X \rightarrow Z$ and $g_2 : X \rightarrow Z$, we have $f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$. In this case, we may denote $f : X \hookrightarrow Y$.²

We call f an *epimorphism* when, for any set Z and any pair of functions $g_1 : Y \rightarrow Z$ and $g_2 : Y \rightarrow Z$, we have $g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$. In this case, we may denote $f : X \twoheadrightarrow Y$.³

We say that f is an *isomorphism* when it has a two-sided inverse; this means there exists a single function $g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$. In this case, the function g is *unique* for f , and we refer to it as *the inverse* of f , denoting it by f^{-1} .

We now have a very important axiom, which is—as it turns out—equivalent to the infamous “*axiom of choice*” (assuming the other axioms of Zermelo-Fraenkel set theory).

Axiom 7: Trichotomy of Cardinality.

For any sets A and B , we have $|A| < |B|$ and $|A| > |B|$ or $|A| = |B|$.

As a result of this axiom, we have several important theorems (though not all of the following theorems necessarily rely on this axiom).

Theorem 10.1: Cantor-Schröder-Bernstein.

$$\forall X \forall Y ((|X| \leq |Y| \wedge |Y| \leq |X|) \Rightarrow |X| = |Y|).$$
Theorem 10.2.

Every injection has a surjective left-inverse (and is therefore a monomorphism).

Corollary 10.1.

$$\forall X \forall Y (|X| \leq |Y| \Rightarrow |Y| \geq |X|).$$
Theorem 10.3.

Every surjection has an injective right-inverse (and is therefore an epimorphism).

Corollary 10.2.

$$\forall X \forall Y (|X| \geq |Y| \Rightarrow |Y| \leq |X|).$$
Theorem 10.4.

Every bijection has a unique two-sided inverse (and is therefore an isomorphism).

Corollary 10.3.

$$\forall X \forall Y (|X| = |Y| \Rightarrow |Y| = |X|).$$
Theorem 10.5.

$$\forall X \forall Y (X \subseteq Y \Rightarrow |X| \leq |Y|).$$
Corollary 10.4.

$$\forall X (|X| = |X|).$$
Theorem 10.6: Pigeonhole Principle.

Given arbitrary sets A and B , the following hold.

- If $|A| > |B|$, then there are no injective functions from A to B .
- If $|A| < |B|$, then there are no surjective functions from A to B .

Finally, if A and B are both finite with cardinalities $n := |A|$ and $k := |B|$, then, for any function $f : A \rightarrow B$, there exists $b \in B$ such that $|\{a \in A \mid f(a) = b\}| \geq \left\lceil \frac{n}{k} \right\rceil = \left\lfloor \frac{n-1}{k} \right\rfloor + 1$.

²We say that f is *left-invertible* when there exists $g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$. Left-invertible functions are necessarily monomorphisms.

³We say that f is *right-invertible* if there exists $g : Y \rightarrow X$ such that $f \circ g = \text{id}_Y$. Right-invertible functions are necessarily epimorphisms.

10.1 Finite Combinatorics

Definition 10.5: Finiteness.

We say that a set F is *finite* iff there exists $n \in \mathbb{N}$ such that $|F| = |n|$. In such a situation, we define the *cardinality of F* to be the natural number n that F is in bijection with: $|F| := n$.

Lemma 10.3.

Every natural number is a finite set. Further, $|\{z \in \mathbb{N} \mid 1 \leq z \leq n\}| = |n|$ for all $n \in \mathbb{N}$.

Lemma 10.4.

For any finite set A and any set B , if $B \subseteq A$, then B is finite.

Lemma 10.5.

For any finite sets A and B , we have $|A \times B| = |A| \cdot |B|$.

Theorem 10.7.

If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.

Theorem 10.8: Inclusion-Exclusion.

Let $n \in \mathbb{N}_+$ and consider a set $\mathcal{A} = \{A_i \mid i \in \mathbb{N} \wedge 1 \leq i \leq n\}$ consisting only of *finite* sets (i.e., A_i is finite for each $i \in \{j \in \mathbb{N} \mid 1 \leq j \leq n\}$). Then, the following inequality holds.

$$|\cup \mathcal{A}| = \left| \bigcup_{i=1}^n A_i \right| = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq n} |A_{i_1} \cap \dots \cap A_{i_k}| \leq \sum_{i=1}^n |A_i|$$

Theorem 10.9.

For any finite X , we have $|\mathbb{P}(X)| = 2^{|X|}$. This implies finite sets have finite power sets.

Definition 10.6: Binomial Coefficients.

For any natural numbers $n, k \in \mathbb{N}$, we define $\binom{n}{k}$ to be the number of subsets of cardinality k taken from a set of cardinality n . Formally, we define this in the following way.

$$\binom{n}{k} := |\{A \mid A \subseteq n \wedge |A| = k\}|$$

However, this definition is not sensitive to the *identity* of the set the subsets are taken from; all that matters is that ambient set's *size*. Therefore, for any finite set X such that $|X| = n$, we have the following.

$$\binom{n}{k} = |\{A \mid A \subseteq X \wedge |A| = k\}|$$

Lemma 10.6.

The binomial coefficients satisfy the following recurrence. For all $n, k \in \mathbb{N}$, we have $\binom{n}{0} = 1 = \binom{n}{n}$, and $\binom{n}{1} = n$, and $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$.

Lemma 10.7.

For all $n, k \in \mathbb{N}$ such that $k \leq n$, we have $\binom{n}{k} = \binom{n}{n-k}$.

Theorem 10.10: Binomial Theorem.

For any real numbers $x, y \in \mathbb{R}$ and any natural number n , we have the following.

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

10.2 Infinite Combinatorics

Definition 10.7: Infinity.

We say that a set I is *infinite* when I is not finite, meaning $(\forall n \in \mathbb{N})(|I| \neq |n|)$.

Theorem 10.11.

\mathbb{N} is infinite.

Theorem 10.12.

$$|\mathbb{N}| = |\mathbb{N}_+| = |\mathbb{Z}| = |\mathbb{Q}| = |\mathbb{N} \times \mathbb{N}|.$$

Theorem 10.13.

Let X be an infinite set and let Y be a set. Then, the following hold.

- $|X \cup Y| = \max(|X|, |Y|)$.
- $|X \times Y| = \max(|X|, |Y|)$.
- If $2 \leq |Y| \leq |X|$, then $|\{f \mid f : X \rightarrow Y\}| = |\{f \mid f : X \rightarrow \{0, 1\}\}| = |\mathbb{P}(X)|$

Definition 10.8: Countability.

We say a set A is *countable* when $|A| \leq |\mathbb{N}|$; otherwise, we say A is *uncountable*. If a countable set A is also infinite, we call it *countably infinite* and denote its cardinality by the special symbol $\aleph_0 := |A|$.

Lemma 10.8.

Every subset of \mathbb{N} is countable.

Corollary 10.5.

Every finite set is countable.

Theorem 10.14.

If A is a countably infinite set, then $|A| = |\mathbb{N}|$.

Theorem 10.15: Countable Unions of Countable Sets are Countable.

Consider a countable collection $\mathcal{A} := \{A_i \mid i \in \mathbb{N}\}$ such that $|A_i| \leq |\mathbb{N}|$ for each $i \in \mathbb{N}$.

Then, $|\cup \mathcal{A}| = |\cup_{i \in \mathbb{N}} A_i| \leq |\mathbb{N}|$.

Definition 10.9: Dedekind Infinity.

Given a set X , we say that X is *Dedekind finite* iff $\forall W((W \subseteq X \wedge W \neq X) \Rightarrow |W| < |X|)$.

Alternatively, we say X is *Dedekind infinite* iff $\exists W(W \subseteq X \wedge W \neq X \wedge |W| = |X|)$.

Theorem 10.16.

Being finite is equivalent to being Dedekind finite, and being infinite is equivalent to being Dedekind infinite.

Corollary 10.6.

The set of natural numbers is the *smallest infinite set*. Formally, $|\mathbb{N}| \leq |I|$ for any infinite I .

Theorem 10.17: Cantor's Theorem.

For any set X , we have $|X| < |\mathbb{P}(X)|$.