

Quiz 2 ver. 2

Discrete Structures

18th day of September of the year of our Lord 2026

Answer each of the following questions according to their instructions.

Recall the axiom schema of weak induction, reproduced below for an arbitrary wff φ .

$$\left(\varphi(0) \wedge \forall k (\varphi(k) \Rightarrow \varphi(\text{suc}(k))) \right) \Rightarrow \forall n (\varphi(n))$$

1. Suppose that, instead of the traditional axiom schema of weak induction that we have learned about, we had introduced a modified version of “induction” that required the following steps for an arbitrary wff φ .¹

- › First, prove $\varphi(0)$ explicitly.
- › Next, prove that, for all k , if k is even, then $\varphi(k/2) \Rightarrow \varphi(k)$.
- › Finally, prove that, for all k , if k is odd, then $\varphi(k^{-1}/2) \Rightarrow \varphi(k)$.

Is this scheme equivalent to induction? In other words: does fulfilling these requirements for a wff φ actually prove $\forall n(\varphi(n))$?

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From the first line, we can verify $\varphi(0)$ explicitly. Now, since 1 is odd, the third line tells us $\varphi(1^{-1}/2) \Rightarrow \varphi(1)$, which simplifies to $\varphi(0) \Rightarrow \varphi(1)$. Using $\varphi(0)$ and *Modus Ponens*, that gives us $\varphi(1)$. Since 2 is even, the second line tells us $\varphi(2/2) \Rightarrow \varphi(2)$, which simplifies to $\varphi(1) \Rightarrow \varphi(2)$. Again, using our earlier result $\varphi(1)$ and *Modus Ponens*, we obtain $\varphi(2)$. Since 3 is odd, the third line tells us $\varphi(3^{-1}/2) \Rightarrow \varphi(3)$, so *Modus Ponens* again yields $\varphi(3)$. Since 4 is even, the third line commands $\varphi(4/2) \Rightarrow \varphi(4)$, forcing us to reckon with $\varphi(4)$. This pattern continues for every natural number k : when k is even, we have already verified $k/2$, and when k is odd, we have already verified $k^{-1}/2$ —in either case, we are able to derive $\varphi(k)$. Therefore, this scheme proves $\forall n(\varphi(n))$, just like weak induction does; it just goes about that in a different order.

Justify your answer to this question with an informal argument.

¹ This scheme involves concepts like “division by two” and “subtraction by one” that we have not defined, and that may not be well-defined on the natural numbers. You may pretend these concepts work in the way you are familiar with from grade school.

2. Given a wff φ , write a proposition (i.e., a wff with no free variables) in terms of φ that would be equivalent to proving φ for every even natural number.² Express your answer in the same style as the axiom schema of weak induction.³

$$\varphi(0) \wedge \forall k (\varphi(k) \Rightarrow \varphi(k+2))$$

No justification is necessary for the answer to this question.

² You may use expressions like “ $x+1$ ” instead of “ $\text{suc}(x)$ ” if you find that more convenient.

³ What this means is: you should write a conjunction that expresses zero or more “basis steps” and zero or more “inductive steps” in the same way that weak induction has one basis case $\varphi(0)$ and one inductive step $\forall k(\varphi(k) \Rightarrow \varphi(\text{suc}(k)))$.