

Quiz 2 ver. 1

Discrete Structures

18th day of September of the year of our Lord 2026

Answer each of the following questions according to their instructions.

Recall the axiom schema of weak induction, reproduced below for an arbitrary wff φ .

$$(\varphi(0) \wedge \forall k(\varphi(k) \Rightarrow \varphi(\text{suc}(k)))) \Rightarrow \forall n(\varphi(n))$$

1. Suppose that, instead of the traditional axiom schema of weak induction that we have learned about, we had introduced a modified version of “induction” that required the following steps for an arbitrary wff φ .

- › First, prove $\varphi(0)$ explicitly.
- › Next, prove that $\varphi(k) \Rightarrow \varphi(k+3)$ for every natural number k .
- › Finally, prove that $\varphi(k+1) \Rightarrow \varphi(k)$ for every natural number k .

Is this scheme equivalent to induction? In other words: does fulfilling these requirements for a wff φ actually prove $\forall n(\varphi(n))$?

It should be clear that this scheme explicitly proves $\varphi(0)$ and also $\varphi(n+3)$ for every natural number n based on the first two lines in the description above. For example, the second line tells us $\varphi(0) \Rightarrow \varphi(0+3)$, and $\varphi(3) \Rightarrow \varphi(3+3)$, and the first line tells us $\varphi(0)$. Combining $\varphi(0)$ with $\varphi(0) \Rightarrow \varphi(0+3)$ proves $\varphi(3)$, and combining *that* with $\varphi(3) \Rightarrow \varphi(3+3)$ proves $\varphi(6)$, and so on. Further, thanks to the third line, whenever we prove $\varphi(n)$ for any particular n , we can work our way *backwards* and fill in any gaps we had missing. So, using $\varphi(3)$ (from earlier) and $\varphi(2+1) \Rightarrow \varphi(2)$ (from the third line), we get $\varphi(2)$. Using $\varphi(2)$ and $\varphi(1+1) \Rightarrow \varphi(1)$, we get $\varphi(1)$.¹ Since we can prove $\varphi(\cdot)$ for arbitrarily large multiples of 3 using the first two lines, and since we can “backfill” any missing numbers all the way back down to 0 using the third line, we can prove $\varphi(n)$ for every natural number n .

2. Given a wff φ , write a proposition (i.e., a wff with no free variables) in terms of φ that would be equivalent to proving φ for every odd natural number.² Express your answer in the same style as the axiom schema of weak induction.³

$$\varphi(1) \wedge \forall k(\varphi(k) \Rightarrow \varphi(k+2))$$

Justify your answer to this question with an informal argument.

¹ An alternative argument:
From $\varphi(0)$, we get $\forall k(\varphi(0+3k))$.
From $\varphi(1)$, we get $\forall k(\varphi(1+3k))$.
From $\varphi(2)$, we get $\forall k(\varphi(2+3k))$.
The three claims above all follow from the second line. We get $\varphi(0)$ from the first line, and we get $\varphi(1)$ and $\varphi(2)$ from the process we described using the first, second, and third lines together. Since every natural number can be expressed as $3k + \ell$ for some natural numbers k and ℓ , where ℓ is either 0, 1, or 2, this is guaranteed to cover all of the natural numbers. This fact should not be obvious, but it is a very important property of the integers that you may be familiar with. We will talk about it in more detail soon.

No justification is necessary for the answer to this question.

² You may use expressions like “ $x+1$ ” instead of “ $\text{suc}(x)$ ” if you find that more convenient.

³ What this means is: you should write a conjunction that expresses zero or more “basis steps” and zero or more “inductive steps” in the same way that weak induction has one basis case $\varphi(0)$ and one inductive step $\forall k(\varphi(k) \Rightarrow \varphi(\text{suc}(k)))$.