Problem Set 7

Discrete Structures

*Due on the **11th day of November of the year of our Lord 2026** at 11:59 pm*

Justify each of your answers with an argument or a proof as appropriate.

Definition: Graph.
A graph G is determined by a *finite* set of *"vertices"* $V(G)$ and a set of *"edges"* $E(G)$ satisfying $E(G) \subseteq \{p \mid p \subseteq V(G) \wedge \#p = 2\}$.

For every natural number $n \in \mathbb{N}$, we define the *path graph of length n,* denoted P_n, to be the graph whose vertices are $V(P_n) \coloneqq \{x \mid x \in \mathbb{N} \wedge x \leqslant n\}$ and whose edges are $E(P_n) \coloneqq \{\{i, i+1\} \mid i \in \mathbb{N} \wedge i < n\}$. A visualization of the path graph P_3 is below.

Figure 1. A visual depiction of the path graph P_3 on the left and the cycle graph C_3 on the right.

For every $n \in \mathbb{N}$ such that $n \geqslant 3$, we define the *cycle graph of length n,* denoted C_n, by $V(C_n) \coloneqq V(P_{n-1})$ and $E(C_n) \coloneqq E(P_{n-1}) \cup \{\{n-1, 0\}\}$. Above, we depict C_3. ⌟

Definition: Graph Homomorphism.
hom
For any graphs G and H, we define *graph homomorphism from G to H* to be a function $f : V(G) \to V(H)$ such that $(\forall v, w \in V(G))(\{v, w\} \in E(G) \Rightarrow \{f(v), f(w)\} \in E(H))$. When f is a graph homomorphism from G to H, it is common to use the abusive notation $f : G \to H$ to emphasize the fact that f is a structure-preserving map between graphs. An *injective* graph homomorphism is called a *graph monomorphism,*
monic
epic
and a *surjective* graph homomorphism is called a *graph epimorphism.* A *bijective* graph homomorphism is called a *graph isomorphism.*[1][2] ⌟

[1] These are technically not the *definitions* of monomorphism, epimorphism, or isomorphism, but they are equivalent to the definitions, and these are *"proper"* ways to think about them.

[2] When we know that $f : G \to H$ is a monomorphism, we often use the notation $f : G \hookrightarrow H$ to emphasize this fact. Similarly, when we know $f : G \to H$ is an epimorphism, the notation $f : G \twoheadrightarrow H$ is sometimes used.

Definition: Walks and Paths.
Let G be a graph, let $v, w \in V(G)$, and $k \in \mathbb{N}$. We define a *walk in G of length k* to be a graph homomorphism $f : P_k \to G$, and we say this is a *walk from v to w iff* $f(0) = v$ and $f(k) = w$. A *path in G of length k* is a graph monomorphism $p : P_k \hookrightarrow G$, and we say this is a *path from v to w iff* $p(0) = v$ and $p(k) = w$. ⌟

Definition: Circuits and Cycles.
Let G be a graph, and $k \in \mathbb{N} \setminus \{0, 1, 2\}$. A *circuit in G of length k* is a graph homomorphism $f : C_k \to G$. A *cycle in G of length k* is a graph monomorphism $p : C_k \hookrightarrow G$. ⌟

1. Given a set X, we call a subset $Y \subseteq X$ *cofinite in X* if and only if $X \setminus Y$ is finite. Prove that there are infinitely many cofinite subsets of $\mathbb{N}$.

2. Consider the set $\mathcal{S} \coloneqq \{2, 3, 6, 8, 10, 19, 24, 25, 32, 33, 50, 60\}$. Show that, for any subset $X \subseteq \mathcal{S}$ of cardinality $\#X \geqslant 9$, there exist elements $x_1, x_2, x_3 \in X$ such that $x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_3 \neq x_1$ and $x_1 + x_2 + x_3 = 68$.

3. Prove, for every graph G, and for every pair of nodes $v, w \in V(G)$, that if there is a *walk in G* from v to w, then there is a *path in G* from v to w.[3]

[3] HINT: try using *(strong)* induction.