

Problem Set 6

Discrete Structures

Due on the 28th day of October of the year of our Lord 2026 at 11:59 pm

Justify each of your answers with an argument or a proof as appropriate.

Definition: Compatibility.

Let f and g be functions. We say that g is *compatible with f* iff $\text{dom}(g) = \text{cod}(f)$. $\lrcorner$

Definition: Function Composition.

Given arbitrary sets A , B , and C , let $f : A \rightarrow B$ and $g : B \rightarrow C$. The *composition of g with f* is a function $g \circ f : A \rightarrow C$ given by $g \circ f(a) := g(f(a))$ for each $a \in A$. $\lrcorner$

Definition: Identity Functions.

Given an arbitrary set X , the *identity function on X* is the function $\text{id}_X : X \rightarrow X$ given by $\text{id}_X(x) := x$ for each $x \in X$. $\lrcorner$

Axiom 8: Dichotomy of Cardinalities.

$\forall x \forall y (|x| \leq |y| \vee |y| \leq |x|)$. $\lrcorner$

Theorem: Existence of Choice Functions.

For any set X such that $X \neq \emptyset$ and $\forall x (x \in X \Rightarrow x \neq \emptyset)$, a “choice function on X ”¹ exists, meaning: there is a function $\tau : X \rightarrow \cup X$ such that $(\forall x \in X)(\tau(x) \in x)$. $\lrcorner$

As an example application of this theorem, consider $X := \{\{0, 1\}, \{2, 3, 4, 5\}, \{6, 7\}\}$. Since X is nonempty, and since all of the elements of X are nonempty, this theorem says there is a function $\tau : X \rightarrow \{0, 1, 2, 3, 4, 5, 6, 7\}$ with the three properties below.

$$\tau(\{0, 1\}) \in \{0, 1\} \quad \tau(\{2, 3, 4, 5\}) \in \{2, 3, 4, 5\} \quad \tau(\{6, 7\}) \in \{6, 7\}$$

The specific function $\tau_X : X \rightarrow \{0, 1, 2, 3, 4, 5, 6, 7\}$ defined below is such an example.

$$\tau_X(\{0, 1\}) := 0 \quad \tau_X(\{2, 3, 4, 5\}) := 4 \quad \tau_X(\{6, 7\}) := 7$$

- Let's prove that composing bijections produces a bijection.
 - Prove that the composition of any two compatible injections is injective.
 - Prove that the composition of any two compatible surjections is surjective.
- For this problem, let A and B be arbitrary sets such that $A \neq \emptyset$ and $B \neq \emptyset$. We will show every injection has a left-inverse and every surjection has a right-inverse.
 - Prove, for any *injection* $f : A \rightarrow B$, there is $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$.
 - Prove, for any *surjection* $f : A \rightarrow B$, there is $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$.
- For this problem, let A and B be arbitrary sets such that $A \neq \emptyset$ and $B \neq \emptyset$. Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be arbitrary functions pointing in opposite directions.
 - Show that, if f is *injective* and $g \circ f = \text{id}_A$, then g is *surjective*.
 - Show that, if f is *surjective* and $f \circ g = \text{id}_B$, then g is *injective*.

¹ This function $\tau : X \rightarrow \cup X$ is called a “choice function” on X because it makes a “choice” about each element $x \in X$ in its domain—the “choice” being made is picking an element $y \in x$ out of the elements of x to use as the value for $\tau(x)$ (this can be thought of as “picking a representative” for x).

N.B. In order to show two functions f and g are equal, there are three things we must prove.

- $\text{dom}(f) = \text{dom}(g)$
- $\text{cod}(f) = \text{cod}(g)$
- $(\forall x \in \text{dom}(f))(f(x) = g(x))$

This last point is equivalent to showing the set of ordered pairs f is equal to the set of ordered pairs g using the Axiom of Extensionality.