Solution Set 5

Discrete Structures

21st day of October of the year of our Lord 2026

1. a. We will show $\forall x \forall y (\cup\cap(x, y) = x)$.

   Proof. Let x and y be sets, and recall that $(x, y) = \{\{x\}, \{x, y\}\}$ by definition. The intersection over a set, applied to $\{\{x\}, \{x, y\}\}$, is defined as follows.

   $$\cap\Big\{\{x\}, \{x, y\}\Big\} = \Big\{w \;\Big|\; \forall z\Big(z \in \Big\{\{x\}, \{x, y\}\Big\} \Rightarrow w \in z\Big)\Big\}$$

   We can then easily verify the following equivalences for any t:

   $$\begin{aligned} t \in (x, y) &\Leftrightarrow t \in \cap\Big\{\{x\}, \{x, y\}\Big\} \\ &\Leftrightarrow \forall z\Big(z \in \Big\{\{z\}, \{x, y\}\Big\} \Rightarrow t \in z\Big) \\ &\Leftrightarrow \big(t \in \{x\}\big) \wedge \Big(t \in \{x, y\}\Big) \\ &\Leftrightarrow (t = x) \wedge \big((t = x) \vee (t = y)\big) \\ &\Leftrightarrow t = x \\ &\Leftrightarrow t \in \{x\} \end{aligned}$$

   This shows us $\cap(x, y) = \{x\}$. Now, recalling $\cup\{x\} = \{w \mid \exists z(z \in \{x\} \wedge w \in z)\}$ by definition, we can again observe a sequence of equivalences for any t:

   $$\begin{aligned} t \in \cup\cap(x, y) &\Leftrightarrow t \in \cup\cap\Big\{\{x\}, \{x, y\}\Big\} \\ &\Leftrightarrow t \in \cup\{x\} \\ &\Leftrightarrow \exists z\big(z \in \{x\} \wedge t \in z\big) \\ &\Leftrightarrow t \in x \end{aligned}$$

   Therefore, by the *Axiom of Extensionality,* we conclude $\cup\cap(x, y) = x$.

   QED

   b. We will show $\forall x \forall y\Big(x \neq y \Rightarrow \cup\Big(\big(\cup(x, y)\big) \setminus \big(\cap(x, y)\big)\Big) = y\Big)$.

   Proof. Let x and y be sets such that $x \neq y$. Recall from the previous problem that we know $\cap(x, y) = \{x\}$. With similar reasoning, we will be able to see that $\cup(x, y) = \{x, y\}$ by observing the following derivation for any t:

   $$\begin{aligned} t \in \cup(x, y) &\Leftrightarrow t \in \cup\Big\{\{x\}, \{x, y\}\Big\} \\ &\Leftrightarrow t \in \Big\{w \;\Big|\; \exists z\Big(z \in \Big\{\{x\}, \{x, y\}\Big\} \wedge w \in z\Big)\Big\} \\ &\Leftrightarrow \exists z\Big(z \in \Big\{\{x\}, \{x, y\}\Big\} \wedge t \in z\Big) \\ &\Leftrightarrow z \in \{x\} \vee z \in \{x, y\} \\ &\Leftrightarrow z \in \{x, y\} \end{aligned}$$

   Now that we know both $\cup(x, y) = \{x, y\}$ and $\cap(x, y) = \{x\}$, it should be clear that $\cup(x, y) \setminus \cap(x, y) = \{x, y\} \setminus \{x\} = \{y\}$ using our assumption $x \neq y$.

Finally, $\cup\{y\} = \{w \mid \exists z(z \in \{y\} \wedge w \in z)\} = \{w \mid w \in y\} = y$. Therefore, we conclude $\cup((\cup(x,y)) \setminus (\cap(x,y))) = y$ by combining these results together.

QED

2. We will prove $\forall A \forall B \exists P(P = A \times B)$.

Proof. Let A and B be sets. Recall that we have already proven $A \cup B$ exists. We then know $\mathbb{P}(A \cup B)$ exists due to the *Axiom of Power*, and thus $\mathbb{P}(\mathbb{P}(A \cup B))$ exists by the same reasoning. Now, consider the following definition.

$$P \coloneqq \left\{ p \;\middle|\; p \in \mathbb{P}\big(\mathbb{P}(A \cup B)\big) \wedge \exists a \exists b \big(a \in A \wedge b \in B \wedge p = (a,b)\big) \right\}$$

Since $\mathbb{P}(\mathbb{P}(A \cup B))$ exists, as discussed above, we know P exists thanks to the *Axiom Schema of Separation*. We claim that $P = A \times B$, which we will prove by relying on the *Axiom of Extensionality*.

In the forward direction, let $\tau \in P$. By definition, this means $\tau \in \mathbb{P}(\mathbb{P}(A \cup B))$, and that there exist $a \in A$ and $b \in B$ such that $\tau = (a,b)$. Since $\tau = (a,b)$ and $a \in A$ and $b \in B$, we can conclude $\tau \in A \times B$ by definition.

In the reverse direction, let $\pi \in A \times B$, which means by definition that there exist $a \in A$ and $b \in B$ such that $\pi = (a,b)$. In order to show $\pi \in P$, all that's left is to show that $\pi \in \mathbb{P}(\mathbb{P}(A \cup B))$. Recalling the way that ordered pairs are defined, we know $\pi = (a,b) = \{\{a\}, \{a,b\}\}$. Observe the following chain of implications.

$$a \in A \quad \Rightarrow \quad \{a\} \subseteq A \quad \Rightarrow \quad \{a\} \in \mathbb{P}(A)$$

These follow immediately from the definitions of set roster notation, subsets, and power sets. We can observe a similar chain of implications below.

$$\begin{aligned}(a \in A) \wedge (b \in B) &\Rightarrow (a \in A \cup B) \wedge (b \in A \cup B) \\ &\Rightarrow \{a,b\} \subseteq A \cup B \\ &\Rightarrow \{a,b\} \in \mathbb{P}(A \cup B)\end{aligned}$$

Now that we know $\{a\} \in \mathbb{P}(A \cup B)$ and $\{a,b\} \in \mathbb{P}(A \cup B)$, the following is clear.

$$\begin{aligned}\big(\{a\} \in \mathbb{P}(A \cup B)\big) \wedge \big(\{a,b\} \in \mathbb{P}(A \cup B)\big) &\Rightarrow \{\{a\}, \{a,b\}\} \subseteq \mathbb{P}(A \cup B) \\ &\Rightarrow \{\{a\}, \{a,b\}\} \in \mathbb{P}\big(\mathbb{P}(A \cup B)\big) \\ &\Rightarrow (a,b) \in \mathbb{P}\big(\mathbb{P}(A \cup B)\big) \\ &\Rightarrow \pi \in \mathbb{P}\big(\mathbb{P}(A \cup B)\big)\end{aligned}$$

Thus, we have that $\pi \in \mathbb{P}(\mathbb{P}(A \cup B))$. Taking all of this together, we have now shown $\pi \in P$ by definition.

Therefore, by the *Axiom of Extensionality*, we can conclude that $P = A \times B$.

QED

3. a. We will show $\exists I \forall Y \exists! f(f : I \to Y)$.

Proof. Consider $I \coloneqq \varnothing$, and let Y be an arbitrary set. We claim that $f \coloneqq \varnothing$ is a function from $\varnothing$ to Y. To justify this claim, we need to show that $f \subseteq \varnothing \times Y$ and that $(\forall x \in \varnothing)(\exists! y \in Y)((x,y) \in f)$.[1]

For the first proposition, notice $\varnothing \subseteq \varnothing \times Y$ because $\forall x(\varnothing \subseteq x)$. So, $f \subseteq \varnothing \times Y$.

[1] Recall $f(x) = y$ is notation defined to mean $(x,y) \in f$ when f is a function.

For the second, we can expand the statement that we are trying to prove so that the form of the sentence is made more explicit.

$$(\forall x \in \varnothing)(\exists! y \in Y)\big((x,y) \in f\big) \quad \Leftrightarrow \quad \forall x\Big(x \in \varnothing \Rightarrow \exists! y\big(y \in Y \wedge (x,y) \in f\big)\Big)$$

Since $\forall x(x \notin \varnothing)$, the premise of the implication in the formula on the right-hand side above is *false*, which makes the overall implication *true*, and therefore the universally quantified implication above is *vacuously true.*

Now that we know f is a function from $\varnothing$ to Y, we want to show it is the *only* such function. Suppose we have *"another"* function $g : \varnothing \to Y$. Then, by definition, we must have $g \subseteq \varnothing \times Y$. Now, observe the following.

$$\varnothing \times Y = \Big\{p \;\Big|\; \exists x \exists y\big(x \in \varnothing \wedge y \in Y \wedge p = (x,y)\big)\Big\} = \Big\{(x,y) \;\Big|\; x \in \varnothing \wedge y \in Y\Big\} = \varnothing$$

This means $g \subseteq \varnothing$, immediately implying $g = \varnothing = f$. Therefore, since any other function from $\varnothing$ to Y is actually just f, we conclude that f is the *only* function from $\varnothing$ to Y.

QED

b. We will show $\exists T \forall X \exists! f(f : X \to T)$.

Proof. Consider $T = \{\varnothing\}$, and let X be an arbitrary set. We claim that the set $f \coloneqq \{p \mid \exists x(x \in X \wedge p = (x,\varnothing))\}$ is a function from $X \to \{\varnothing\}$. To show this, we will prove that $f \subseteq X \times \{\varnothing\}$ and $(\forall x \in X)(\exists! y \in \{\varnothing\})((x,y) \in f)$.

First, take an arbitrary element $p \in f$, so that $p = (x,\varnothing)$ for some $x \in X$ by definition. Since $\varnothing \in \{\varnothing\}$, we immediately have that $(x,\varnothing) \in X \times \{\varnothing\}$. This proves $f \subseteq X \times \{\varnothing\}$.

Second, let $x \in X$. Notice that $\varnothing \in \{\varnothing\}$, so $(x,\varnothing) \in f$ by definition, proving the existential claim $(\exists y \in \{\varnothing\})((x,y) \in f)$. To show uniqueness, suppose there were some *"other"* output t for x, satisfying $t \in \{\varnothing\}$ and $(x,t) \in f$. Then, the fact that $t \in \{\varnothing\}$ implies $t = \varnothing$. This proves uniqueness of the output under f for any given input.

We will now show that f is the *only* function from X to $\{\varnothing\}$. Suppose we had some *"other"* function $g : X \to \{\varnothing\}$. Then, because g is a function, we know $(\forall x \in X)(\exists! y \in \{\varnothing\})(g(x) = y)$. Since $\forall y(y \in \{\varnothing\} \Leftrightarrow y = \varnothing)$, we can make the following observation for every $x \in X$:

$$\big(\exists! y \in \{\varnothing\}\big)\big(g(x) = y\big) \quad \Leftrightarrow \quad g(x) = \varnothing \quad \Leftrightarrow \quad g(x) = f(x)$$

Therefore, $(\forall x \in X)(g(x) = f(x))$. Since g has the same domain, codomain, and mapping as f, these functions are equal: $f = g$. Therefore, f is the *only* function from X to $\varnothing$.

QED