

Problem Set 5

Discrete Structures

Due on the **21st day of October of the year of our Lord 2026** at 11:59 pm

Justify each of your answers with an argument or a proof as appropriate.

Answers presented without proof or justification are worse than incorrect.

Axiom 0: Infinity.

There exists a set—which we will denote $\mathbb{N}$ —with the following properties.

1. $\emptyset \in \mathbb{N} \wedge \forall n(n \in \mathbb{N} \Rightarrow n \cup \{n\} \in \mathbb{N})$
2. $\forall M((\emptyset \in M \wedge \forall m(m \in M \Rightarrow m \cup \{m\} \in M)) \Rightarrow \mathbb{N} \subseteq M)$

Axiom 1: Extensionality.

$\forall x \forall y (x = y \Leftrightarrow \forall z (z \in x \Leftrightarrow z \in y))$

Axiom 2: Pairing.

$\forall x \forall y \exists z (z = \{x, y\})$

Axiom 3: Union.

$\forall x \exists y (y = \cup x)$

Axiom 4: Schema of Separation.

For any wff φ with at most one free variable, we have $\forall x \exists y (y = \{z \mid z \in x \wedge \varphi(z)\})$.

Axiom 5: Power.

$\forall x \exists y (y = \{z \mid z \subseteq x\})$

Axiom 6: Regularity.

$\forall x (x \neq \emptyset \Rightarrow \exists y (y \in x \wedge x \cap y = \emptyset))$

(x, y)

1. Given two arbitrary sets x and y , we define the **ordered pair** between them to be the set $(x, y) := \{\{x\}, \{x, y\}\}$. This is a “good” definition for the idea of an *ordered* pair because $(x, y) = (a, b) \Leftrightarrow ((x = a) \wedge (y = b))$ for any choice of x, y, a , and b .

- a. To show how we can recover the first coordinate, prove $\forall x \forall y (\cup \cap (x, y) = x)$.
- b. For the second, prove $\forall x \forall y (x \neq y \Rightarrow \cup((\cup(x, y)) \setminus (\cap(x, y))) = y)$.

$A \times B$

2. Given two arbitrary sets A and B , we define the **Cartesian product** between them to be the set $A \times B := \{p \mid \exists a \exists b (a \in A \wedge b \in B \wedge p = (a, b))\}$. This definition can be expressed with more concise notation as $A \times B := \{(a, b) \mid a \in A \wedge b \in B\}$.¹ Prove that $\forall A \forall B \exists P (P = A \times B)$.

$f : X \rightarrow Y$

$\text{dom}(f)$

$\text{cod}(f)$

3. Given two arbitrary sets X and Y , we say by definition that a set f is a **function from X to Y** iff $f \subseteq X \times Y$ and $(\forall x \in X)(\exists! y \in Y)((x, y) \in f)$. In such a situation, we use the notation $f : X \rightarrow Y$ to mean that “ f is a function from X to Y .” We then define the *domain* of f as $\text{dom}(f) := X$ and the *codomain* of f as $\text{cod}(f) := Y$. When $x \in X$ and $y \in Y$, we define the functional notation $f(x) = y \Leftrightarrow (x, y) \in f$.
 - a. Does there exist I such that, for every Y , there is a *unique* function $f : I \rightarrow Y$?
 - b. Does there exist T such that, for every X , there is a *unique* function $f : X \rightarrow T$?

In this problem, you may take for granted that $\forall x \forall y (\cup \{x, y\} = x \cup y)$ and $\forall x \forall y (\cap \{x, y\} = x \cap y)$.

¹ You can use either version of the definition. The first version is more explicit (more “basic”), so if you have trouble understanding *what it means to be an element of $\{(a, b) \mid a \in A \wedge b \in B\}$ formally*, then you should stick to the first definition, which explicitly tells you what it means for an arbitrary object p to be an element of $A \times B$.

You must justify your answers to these questions with a *proof*!