Solution Set 4

Discrete Structures

14th day of October of the year of our Lord 2026

1. a. We will prove $\forall x(\mathfrak{suc}(x) \neq 0)$.

   Proof. Let x be a set, and notice that $x \in \{x\}$,[1] which implies $x \in x \vee x \in \{x\}$. This means precisely that $x \in x \cup \{x\}$ by definition,[2] and so $x \in \mathfrak{suc}(x)$. Now, recalling that $\forall w(w \notin \varnothing)$, we deduce $x \notin \varnothing$. Since $x \in \mathfrak{suc}(x)$ but $x \notin \varnothing$, we conclude that $\mathfrak{suc}(x) \neq \varnothing$ by the *Axiom of Extensionality.*

   QED

   [1] This follows from the definition of *"set roster notation."*

   [2] The definition of the union between two sets A and B tells us, for any z, that $z \in A \cup B \Leftrightarrow z \in A \vee z \in B$.

   b. We will prove $\forall x \exists y(y = \mathfrak{suc}(x))$.

   Proof. Let x be a set. By the *Axiom of Pairing,* we know there exists some p such that $p = \{x, x\}$, and we know $\{x, x\} = \{x\}$ by the *Axiom of Extensionality.* Then, since x and p are both sets, we know that there exists a set s such that $s = x \cup p$, which means precisely that $s = x \cup \{x\} = \mathfrak{suc}(x)$, as desired.

   QED

   c. We take for granted the theorem[3] that $\neg\exists x \exists y(x \in y \wedge y \in x)$.
   We will prove $\forall x \forall y(\mathfrak{suc}(x) = \mathfrak{suc}(y) \Rightarrow x = y)$.

   [3] The *Axiom of Regularity* is required to prove this theorem.

   Proof. Let x and y be sets such that $\mathfrak{suc}(x) = \mathfrak{suc}(y)$.

   Notice that $x \in \{x\}$, implying $x \in x \vee x \in \{x\}$ so that $x \in x \cup \{x\}$ by definition. Since $x \cup \{x\} = \mathfrak{suc}(x) = \mathfrak{suc}(y) = y \cup \{y\}$, this implies $x \in y \cup \{y\}$, from which we can see $x \in y \vee x \in \{y\}$. We follow a similar argument with y below.

   $$\big(y \in \{y\}\big) \Rightarrow \big(y \in y \vee y \in \{y\}\big) \Rightarrow \big(y \in y \cup \{y\}\big) \Rightarrow \big(y \in x \cup \{x\}\big) \Rightarrow \big(y \in x \vee y \in \{x\}\big)$$

   Thus, we know $(x \in y \vee x \in \{y\}) \wedge (y \in x \vee y \in \{x\})$. This proposition is logically equivalent to the statement below.

   $$(x \in y \wedge y \in x) \vee \big(x \in y \wedge y \in \{x\}\big) \vee \big(x \in \{y\} \wedge y \in x\big) \vee \big(x \in \{y\} \wedge y \in \{x\}\big)$$

   Since we were given $\neg\exists a \exists b(a \in b \wedge b \in a)$, we know $\neg(x \in y \wedge y \in x)$, eliminating the first clause from the disjuntion above and leaving three cases.

   Case 1:
   Suppose $x \in y \wedge y \in \{x\}$. From $y \in \{x\}$, we infer $y = x$, and we are done.

   Case 2:
   Suppose $x \in \{y\} \wedge y \in x$. From $x \in \{y\}$, we infer $x = y$, and we are done.

   Case 3:
   Suppose $x \in \{y\} \wedge y \in \{x\}$. From $x \in \{y\}$, we get $x = y$, and we are done.

   Having reached $x = y$ in each case, we conclude that $x = y$.

   QED

   d. We take for granted the schema of weak induction adapted for $\mathbb{N}$ in set theory.
   We will prove that every natural number is an $\in$-transitive set—which formally means $(\forall x \in \mathbb{N})(\forall y \forall z((y \in x \wedge z \in y) \Rightarrow z \in x))$—by weak induction.

Proof.

Basis Step:
Suppose, towards a contradiction, that $\varnothing$ is *not* $\in$-transitive. That means, by definition, that there exists some y and z such that $y \in \varnothing \wedge z \in y$ but $z \notin \varnothing$. However, knowing that $y \in \varnothing$ clearly contradicts $\forall w(w \notin \varnothing)$. ↯
Therefore, $\varnothing$ is $\in$-transitive.

Inductive Step:
Let $k \in \mathbb{N}$, and assume that k is $\in$-transitive.[1] In order to show that $\mathfrak{suc}(k)$ is $\in$-transitive, let y and z be sets such that $y \in \mathfrak{suc}(k)$ and $z \in y$. Our goal is to show that $z \in \mathfrak{suc}(k)$. Recalling that $\mathfrak{suc}(k) = k \cup \{k\}$, we can infer from $y \in \mathfrak{suc}(k)$ that $y \in k \cup \{k\}$, which means $y \in k \vee y \in \{k\}$ by definition. We now take two cases based on this disjunction.

[1] This is our *inductive hypothesis.*

Case 1:
Suppose $y \in k$. Since $y \in k$ and $z \in y$, we can see $z \in k$ by the *inductive hypothesis.* It then follows that $z \in k \vee z \in \{k\}$, so that $z \in k \cup \{k\}$.

Case 2:
Suppose $y \in \{k\}$. This immediately implies $y = k$. Recalling $z \in y$, we can then see $z \in k$. Thus, $z \in k \vee z \in \{k\}$, and therefore $z \in k \cup \{k\}$.

As a result, we now know $z \in k \cup \{k\}$, which means $z \in \mathfrak{suc}(k)$ as desired. Since y and z were arbitrary, we conclude that $\mathfrak{suc}(k)$ is $\in$-transitive.

Therefore, by the theorem of weak induction on $\mathbb{N}$, we conclude that every natural number is an $\in$-transitive set.

QED

2. Using weak induction, we will prove, for every $n \in \mathbb{N}$, and for every list $x_1, ..., x_n$ of n-many arbitrary objects, that the set $\{z \mid \exists i(i \in \mathbb{N} \wedge 1 \leqslant i \leqslant n \wedge z = x_i)\}$ containing those objects as elements exists.

Proof.

Basis Step:
Letting $n := 0$, take an arbitrary list of objects $x_1, ..., x_0$.[2] In this portion of our proof, whenever we make reference to x_j for some natural number j in the range $1 \leqslant j \leqslant 0$, it will refer to the corresponding object at position j in $x_1, ..., x_0$. Note: as the notation would suggest, there are not actually any objects in this list since there are no natural numbers j that satisfy $1 \leqslant j \wedge j \leqslant 0$. Now, let $X := \{z \mid \exists i(i \in \mathbb{N} \wedge 1 \leqslant i \leqslant n \wedge z = x_i)\}$. Notice that, if there were some $x \in X$, then there would exist some $j \in \mathbb{N}$ such that $1 \leqslant j$ and $j \leqslant 0$ and $x = x_j$ by definition, which would imply $1 \leqslant 0$, which contradicts $0 < 1$. Therefore, there are no elements in X, so $X = \varnothing$. Since we know that $\varnothing$ exists, we conclude that $\{z \mid \exists i(i \in \mathbb{N} \wedge 1 \leqslant i \leqslant n \wedge z = x_i)\}$ exists.

[2] Notice: this notation is fundamentally informal. Somehow, we need to communicate the idea that we have a *"list of objects"* indexed by natural numbers starting at 1 and ending at n, where n is our chosen natural number. But, regardless of how we attempt to convey that, we won't be able to say anything more formal than this without assuming the conclusion.

Inductive Step:
Let $k \in \mathbb{N}$, and assume, for any list of k-many arbitrary objects $x_1, ..., x_k$, that the collection $\{z \mid \exists i(i \in \mathbb{N} \wedge 1 \leqslant i \leqslant k \wedge z = x_i)\}$ exists.[3] Now, let $x_1, ..., x_{k+1}$ be a list of $(k+1)$-many objects. By the *inductive hypothesis*, there exists a set X such that $X = \{z \mid \exists i(i \in \mathbb{N} \wedge 1 \leqslant i \leqslant k \wedge z = x_i)\}$. By

[3] This is our *inductive hypothesis.*

the *Axiom of Pairing* and the *Axiom of Extensionality,* we know that the set $\{x_{k+1}\}$ exists, and therefore that $X \cup \{x_{k+1}\}$ exists. Our claim here is that $X \cup \{x_{k+1}\} = \{z \mid \exists i(i \in \mathbb{N} \wedge 1 \leqslant i \leqslant k+1 \wedge z = x_i)\}$, which we will prove by showing those two collections have the same elements. Let x be a set.

First, let $x \in X \cup \{x_{k+1}\}$. This means $x \in X$ or $x \in \{x_{k+1}\}$. We take two cases.

Case 1:
Suppose $x \in X$. This means there exists $i \in \mathbb{N}$ such that $1 \leqslant i \leqslant k$ and $x = x_i$. Since $1 \leqslant i \leqslant k$ and $k < k+1$, we know $1 \leqslant i \leqslant k+1$, and therefore $x \in \{z \mid \exists i(i \in \mathbb{N} \wedge 1 \leqslant i \leqslant k+1 \wedge z = x_i)\}$ by definition.

Case 2:
Suppose that $x \in \{x_{k+1}\}$, which implies $x = x_{k+1}$. Since $k+1 \in \mathbb{N}$ and $1 \leqslant k+1 \leqslant k+1$, we have $x \in \{z \mid \exists i(i \in \mathbb{N} \wedge 1 \leqslant i \leqslant k+1 \wedge z = x_i)\}$.

Therefore, $x \in \{z \mid \exists i(i \in \mathbb{N} \wedge 1 \leqslant i \leqslant k+1 \wedge z = x_i)\}$, and this half is done.

Second, let $x \in \{z \mid \exists i(i \in \mathbb{N} \wedge 1 \leqslant i \leqslant k+1 \wedge z = x_i)\}$, which means there exists some $j \in \mathbb{N}$ such that $1 \leqslant j \leqslant k+1$ and $x = x_j$. We can see that either $1 \leqslant j \leqslant k$ or $j = k+1$, so we take two cases.

Case 1:
Suppose $1 \leqslant j \leqslant k$. Then, $x \in X$ by definition, so $x \in X \vee x \in \{x_{k+1}\}$. This then implies $x \in X \cup \{x_{k+1}\}$ by definition.

Case 2:
Suppose $j = k+1$, so that $x = x_{k+1}$. Clearly, we then see $x \in \{x_{k+1}\}$, so that $x \in X \vee x \in \{x_{k+1}\}$. This then shows $x \in X \cup \{x_{k+1}\}$ by definition.

Therefore, $x \in X \cup \{x_{k+1}\}$, and this half is also done.

We thus conclude $X \cup \{x_{k+1}\} = \{z \mid \exists i(i \in \mathbb{N} \wedge 1 \leqslant i \leqslant k+1 \wedge z = x_i)\}$ by the *Axiom of Extensionality,* and this finishes our inductive step.

Therefore, by the theorem of weak induction on $\mathbb{N}$, we have shown that any finite list of objects from our universe of discourse can be placed into a set.

QED