Problem Set 4

Discrete Structures

Due on the ***14th day of October of the year of our Lord 2026*** *at 11:59 pm*

Justify each of your answers with an argument or a proof as appropriate. Answers presented without proof or justification are worse than incorrect.

This first page lists the basic definitions and axioms—and the one theorem—we have studied so far. The second page contains the problems for this assignment.

Definition: Set Roster Notation.
Given a finite list $x_1, ..., x_n$ of n objects—where n is a natural number—we introduce the notation $\{x_1, ..., x_n\}$ to represent the set containing all (and only) the elements $x_1, ..., x_n$. We formally describe this notation by defining what it means for an arbitrary set z to be an element of $\{x_1, ..., x_n\}$ below.

$$z \in \{x_1, ..., x_n\} \quad :\Leftrightarrow \quad (z = x_1 \vee \cdots \vee z = x_n)$$

We interpret $z = x_1 \vee \cdots \vee z = x_n$ to be *false* when $n = 0$ in the above definition.[1] ⌟

[1] Notice that this—the interpretation of an *empty iterated disjunction*—is similar to the situation for interpreting empty iterated sums and empty iterated products because $\bot$ behaves like an *"identity element"* with respect to the $\vee$ operation on Boolean expressions.

Definition: Set Comprehension Notation.
Given a *wff* φ, we introduce notation to represent the collection of all objects in the universe that make the formula $\varphi(\circ)$ *true.* ⌟

Definition: Empty Set.
We say that an arbitrary set x *is empty* $:\Leftrightarrow \forall y(y \notin x)$. It turns out that there *exists* a set with this property, and this set is *unique.* We call this set *"the empty set"* and use the symbol $\varnothing := \{z \mid z \neq z\}$ to denote it. ⌟

Idea: ("Axiom 0").
The set of all natural numbers exists. We use the symbol $\mathbb{N}$ to denote this set. *Informally speaking,* this set would look something like $\mathbb{N} = \{0, 1, 2, ...\}$. ⌟

Axiom 1: Extensionality.
$\forall x \forall y(x = y \Leftrightarrow \forall z(z \in x \Leftrightarrow z \in y))$

Axiom 2: Pairing.
$\forall x \forall y \exists z(z = \{x, y\})$

Axiom 3: Union.
$\forall x \exists y(y = \cup x)$

Axiom 4: Schema of Separation.
For any *wff* φ with at most one free variable, we have $\forall x \exists y(y = \{z \mid z \in x \wedge \varphi(z)\})$.

Theorem: Existence and Uniqueness of the Empty Set.
$\exists E(\forall w(w \notin E) \wedge \forall x(\forall w(w \notin x) \Rightarrow x = E))$.

Theorem: Existence of Finite Unions.
$\forall x \forall y \exists z(z = x \cup y)$

Theorem: Existence of Finite Intersections.
$\forall x \forall y \exists z(z = x \cap y)$

1. For this problem, we will attempt to recover arithmetic within set theory by providing an interpretation of 0 and $\mathfrak{suc}(\circlearrowleft)$ within set theory and then proving that the axioms of arithmetic are *true* in set theory.[1]

   Given an arbitrary set x, the *successor of* x is defined by $\mathfrak{suc}(x) \coloneqq x \cup \{x\}$. We define the symbol 0 to represent the empty set $0 \coloneqq \varnothing$, and we define the symbols $1, 2, 3, \ldots$ to represent successors of 0 just as we did in the theory of arithmetic.[2] This gives us a way to express what a *"natural number"* is in set theory—we say that an arbitrary set x is a *"natural number" iff* $x = 0 = \varnothing$ or $x = \mathfrak{suc}(y) = y \cup \{y\}$ for some *natural number* y.

   a. Prove $\forall x(\mathfrak{suc}(x) \neq 0)$.

   b. Prove $\forall x \exists y(y = \mathfrak{suc}(x))$.

   c. For this problem, please assume that $\neg\exists x \exists y(x \in y \wedge y \in x)$ has been proven. Prove $\forall x \forall y(\mathfrak{suc}(x) = \mathfrak{suc}(y) \Rightarrow x = y)$.

   d. Now, suppose we knew the schema for *weak induction* worked with our new understanding of natural numbers—more precisely, for any *wff* φ with at most one free variable, assume the following weak induction schema holds.

$$\Big(\varphi(0) \wedge \forall k\Big(k \in \mathbb{N} \Rightarrow \big(\varphi(k) \Rightarrow \varphi\big(\mathfrak{suc}(k)\big)\big)\Big)\Big) \Rightarrow \forall n\big(n \in \mathbb{N} \Rightarrow \varphi(n)\big)$$

   Given a set x, we say x is a $\in$*-transitive set* $:\Leftrightarrow \forall y \forall z((y \in x \wedge z \in y) \Rightarrow z \in x)$. Use induction to prove that every natural number is an $\in$-transitive set.

2. For this problem, we will momentarily pretend that we *"understand"* what a natural number is, and we will assume that we can perform (any form of) induction in the manner that we did when we were studying the theory of arithmetic.

   Let $n \in \mathbb{N}$, and suppose you are given a list of n distinct objects $x_1, \ldots, x_n$ from our universe. Use induction[3] to prove[4] that $\{x_i \mid i \in \mathbb{N} \wedge 1 \leqslant i \leqslant n\}$ exists.[5]

[1] Notice that the first axiom of arithmetic—$\exists x(x = 0)$—is already a theorem if we define $0 \coloneqq \varnothing$. See the first page; we will prove this in lecture using the *axiom schema of separation*

[2] To be explicit, this means $1 \coloneqq \mathfrak{suc}(0)$, $2 \coloneqq \mathfrak{suc}(1)$, $3 \coloneqq \mathfrak{suc}(2)$, and so on, with the understanding that 0 now means $\varnothing$ and $\mathfrak{suc}(x)$ now means $x \cup \{x\}$.

[3] Weak induction is easiest here.

[4] You can rely on any axioms or theorems you *"remember"* from the theory of arithmetic without justification, but your proof should be based on the axioms and theorems of set theory.

[5] HINT: A formal way to express the statement you are trying to prove is: For every natural number n, for every choice of n objects $x_1, \ldots, x_n$, there exists z equal to the set below.

$$\Big\{w \;\Big|\; \exists i\big(i \in \mathbb{N} \wedge 1 \leqslant i \leqslant n \wedge w = x_i\big)\Big\}$$