

Problem Set 3

Discrete Structures

Due on the 5th day of October of the year of our Lord 2026 at 11:59 pm

Justify each of your answers with an argument or a proof as appropriate.

Answers presented without proof or justification are worse than incorrect.

For this problem set, *unless otherwise stated*, you may rely on any of the theory we have established up through and including LN07. In particular, this means you may rely on the theorems that establish *addition* and *multiplication* as operations on natural numbers that are associative and commutative with respective identity elements 0 and 1. You may also rely on distributivity of multiplication over addition, the fact that 0 is the multiplicative annihilator, the fact that there are no nonzero zero-divisors, and all of the properties that make $\leq$ a linear order on the natural numbers.¹

From this point forward, for any natural number x , you may treat $\text{suc}(x)$ the same as $x + 1$ without justification.² You may also take for granted “*arithmetic*” facts involving *addition* and *multiplication* about *specific* natural numbers without justification.³

Given a natural number n , we will use $\mathcal{F}(n)$ to denote the n^{th} **Fibonacci number**. Given two natural numbers x and y , we will use $\mathcal{B}(x, y)$ to denote a term in a particular kind of two-dimensional sequence. These are defined recursively below.⁴

$$\begin{aligned}\mathcal{F}(0) &:= 0 & \mathcal{B}(0, 0) &:= 1 \\ \mathcal{F}(1) &:= 1 & \mathcal{B}(0, y + 1) &:= 0 \\ \mathcal{F}(n + 2) &:= \mathcal{F}(n + 1) + \mathcal{F}(n) & \mathcal{B}(x + 1, 0) &:= \mathcal{B}(x, 1) \\ & & \mathcal{B}(x + 1, y + 1) &:= \mathcal{B}(x, y) + \mathcal{B}(x, y + 2)\end{aligned}$$

1. Prove that $1 + \sum_{i=0}^n 2^i = 2^{n+1}$ for every natural number n .
2. Prove that $\sum_{i=0}^n \mathcal{F}(i)^2 = \mathcal{F}(n) \cdot \mathcal{F}(n + 1)$ for every natural number n .
3. Recall the definition of the two-dimensional sequence $\mathcal{B}(\odot, \odot)$ from above.
 - a. Prove that $\forall x \forall y (x < y \Rightarrow \mathcal{B}(x, y) = 0)$.
 - b. Show that $\mathcal{B}(x, x) = 1$ for every natural number x .
 - c. We define the sequence of $\mathcal{C}(n)$ of **Catalan numbers** by $\mathcal{C}(n) := \mathcal{B}(2 \cdot n, 0)$ for every natural number n . Prove that the Catalan numbers satisfy $\forall n (1 \leq \mathcal{C}(n))$.
4. Consider the recurrence relations $K(\odot)$ and $M(\odot)$ defined below.⁵

$$\begin{aligned}K(0) &:= 0 & M(0) &:= 0 \\ K(1) &:= 1 & M(1) &:= 1 \\ K(2 \cdot n) &:= 3 \cdot K(n) + n & M(2 \cdot n) &:= 4 \cdot M(n) + n\end{aligned}$$

- a. Find a closed form for $K(n)$ and prove that your answer is correct.⁶
- b. Find a closed form for $M(n)$ and prove that your answer is correct.⁷

¹ Note that, since a definition does not require justification and does not make an assertion, you may *always* rely on definitions we have studied.

² E.g., in a proof by induction, you may now prove $\forall k (\varphi(k) \Rightarrow \varphi(k + 1))$ in the inductive step instead of proving $\forall k (\varphi(k) \Rightarrow \varphi(\text{suc}(k)))$ without further justification.

³ E.g., you may rely on theorems like $1 + 1 = 2$, and $2 \cdot 3 = 6$, and so on without further justification. However, if you want to assert $2 \cdot x = x \cdot 2$, you will have to justify this with the commutativity of multiplication.

⁴ It might be helpful to compute the first few values of $\mathcal{B}(x, y)$ by hand in a grid, with x going horizontally and y going vertically. Here are the first few.

$x \backslash y$	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	1	0	1					
3	0	2	0	1				
4	2	0	3	0	1			
5	0	5	0	4	0	1		
6	5	0	9	0	5	0	1	
7	0	14	0	14	0	6	0	1

⁵ These correspond to recurrences for real algorithms describing two different approaches for computing the product $x \cdot y$ of two integers x and y .

⁶ For appropriate values of n .

⁷ For appropriate values of n .