

Solution Set 2

Discrete Structures

21st day of September of the year of our Lord 2026

1. a. We will prove that $\exists x(x = 2)$.

Proof. We know from *Axiom 0*¹ that there exists a natural number x such that $x = 0$.² From *Axiom 2*,³ we then know that there exists a natural number y such that $y = \text{succ}(x)$.⁴ Applying *Axiom 2* again, we also know that there exists a natural number z such that $z = \text{succ}(y)$. We can now compute the following.

$$\begin{aligned} z = \text{succ}(y) &\Leftrightarrow z = \text{succ}(\text{succ}(x)) && \text{because } y = \text{succ}(x) \\ &\Leftrightarrow z = \text{succ}(\text{succ}(0)) && \text{because } x = \text{succ}(0) \\ &\Leftrightarrow z = \text{succ}(1) && \text{because } 1 = \text{succ}(0) \text{ by definition of } 1 \\ &\Leftrightarrow z = 2 && \text{because } 2 = \text{succ}(1) \text{ by definition of } 2 \end{aligned}$$

This shows us that $z = 2$. Therefore, we can conclude that $\exists t(t = 2)$.⁵

QED

- b. We will prove that $\forall x(\text{succ}(x) = x + 1)$.

Proof. Let x be an arbitrary natural number. We can now simply observe.

$$\begin{aligned} x + 1 &= x + \text{succ}(0) && \text{because } 1 = \text{succ}(0) \text{ by definition of } 1 \\ &= \text{succ}(x + 0) && \text{by the definition of addition} \\ &= \text{succ}(x) && \text{since } x + 0 = x \text{ by the definition of addition} \end{aligned}$$

Thus, we have $\text{succ}(x) = x + 1$. We can therefore conclude $\forall t(\text{succ}(t) = t + 1)$.⁶

QED

2. We will show that $\forall x \forall y(\text{succ}(x) + y = \text{succ}(x + y))$.

Proof. Let x be an arbitrary natural number. We will use induction to show that $\forall y(\text{succ}(x) + y = \text{succ}(x + y))$. Although it's not required, we will write at the beginning of each step a sentence describing what it is we are going to prove.⁷

Basis Step:

We need to show that $\text{succ}(x) + 0 = \text{succ}(x + 0)$. Towards that end, first notice that $\text{succ}(x) + 0 = \text{succ}(x)$ by the definition of addition. Similarly, it should be clear that $\text{succ}(x) = \text{succ}(x + 0)$ because we know $x + 0 = x$ by the definition of addition. Combining our two results, we conclude $\text{succ}(x) + 0 = \text{succ}(x + 0)$.

Inductive Step:

In this step, we need to prove the following statement.

$$\forall k \left((\text{succ}(x) + k = \text{succ}(x + k)) \Rightarrow (\text{succ}(x) + \text{succ}(k) = \text{succ}(x + \text{succ}(k))) \right)$$

Let k be an arbitrary natural number. Assume $\text{succ}(x) + k = \text{succ}(x + k)$.⁸ We will now show that $\text{succ}(x) + \text{succ}(k) = \text{succ}(x + \text{succ}(k))$. Observe the following.

¹ Recall that *Axiom 0* states $\exists x(x = 0)$.

² We have implicitly applied *Existential Elimination* to *Axiom 0*. Since that is a logical rule of inference, we don't bother stating that justification in the proof, and we can skip writing that step if we like.

³ Recall *Axiom 2* says $\forall x \exists y(y = \text{succ}(x))$.

⁴ We have implicitly applied *Universal Elimination* with $x = 0$, followed by *Existential Elimination*, on *Axiom 2*.

⁵ This is an application of *Existential Introduction*, which we don't bother stating since it is a rule of inference.

⁶ This is an instance of *Universal Introduction* applied to $\text{succ}(x) = x + 1$.

⁷ The *wff* we want to show universally is $\varphi(n) := \text{"succ}(x) + n = \text{succ}(x + n)\text{"}$.

⁸ This is our *inductive hypothesis*.

$$\begin{aligned}
\text{suc}(x) + \text{suc}(k) &= \text{suc}(\text{suc}(x) + k) && \text{by the definition of addition} \\
&= \text{suc}(\text{suc}(x + k)) && \text{by the inductive hypothesis} \\
&= \text{suc}(x + \text{suc}(k)) && \text{by the definition of addition}
\end{aligned}$$

As a result, $\text{suc}(x) + \text{suc}(k) = \text{suc}(x + \text{suc}(k))$, ending our inductive step.⁹

Therefore, we conclude $\forall n \forall m (\text{suc}(n) + m = \text{suc}(n + m))$.

⁹ We are leaving the final applications of *Implication Introduction* and *Universal Introduction* implicit and unstated.

QED

3. Let φ be an arbitrary wff. We prove $\forall x(\varphi(x)) \Rightarrow (\varphi(0) \wedge \forall k(\varphi(k) \Rightarrow \varphi(\text{suc}(k))))$.

Proof. We begin by assuming $\forall x(\varphi(x))$. We now need to show that $\varphi(0)$, and also that $\forall k(\varphi(k) \Rightarrow \varphi(\text{suc}(k)))$. We will do this in two steps.

First, recall that 0 is a natural number by *Axiom o*. This implies $\varphi(0)$ because of the fact that we know $\forall x(\varphi(x))$ by assumption.

Second, let n be an arbitrary natural number. Assume that $\varphi(n)$ holds. We now need to prove that $\varphi(\text{suc}(n))$ also holds. Notice, since n is a natural number, that $\text{suc}(n)$ is also a natural number by *Axiom 2*. Thus, because we assumed $\forall x(\varphi(x))$, it follows that $\varphi(\text{suc}(n))$. It should therefore be clear that $\forall k(\varphi(k) \Rightarrow \varphi(\text{suc}(k)))$.

Having shown both $\varphi(0)$ and $\forall k(\varphi(k) \Rightarrow \varphi(\text{suc}(k)))$, we conclude with the observation that $\forall x(\varphi(x)) \Rightarrow (\varphi(0) \wedge \forall k(\varphi(k) \Rightarrow \varphi(\text{suc}(k))))$ as was required.

QED

4. There is really only one flaw in the argument, and it takes place in the inductive step.¹⁰ To better understand this, let's try to write down what the form of the argument looks like. For simplicity, we will refer to a collection of people as *monochromatic* if all of the people in that collection have the same eye color. Let's define the unary φ , which takes one natural number argument, as below.

$$\varphi(n) := \text{"Every collection of } n \text{ many people is monochromatic."}$$

Note that the argument takes care to justify that $\varphi(0)$ is *true* because an *empty collection* of people must necessarily be monochromatic.¹¹ The argument presented is an attempt at induction, so the form of the argument is as follows.

$$\varphi(0) \wedge \varphi(1) \wedge \forall k(\varphi(k) \Rightarrow \varphi(k + 1))$$

Adding an additional basis step $\varphi(1)$, as done above, is equivalent to arguing $\varphi(0) \wedge \forall k(\varphi(k) \Rightarrow \varphi(k + 1))$. Let's very quickly try to see why below.

In the forward direction, if we have only $\varphi(0)$ and $\forall k(\varphi(k) \Rightarrow \varphi(k + 1))$, then we know $\varphi(0) \Rightarrow \varphi(1)$ by *Universal Elimination* on our second premise. Combining this with $\varphi(0)$ and applying *Modus Ponens*, we now know $\varphi(1)$.

In the backwards direction, if we know $\varphi(0)$, and $\varphi(1)$, and $\forall k(\varphi(k) \Rightarrow \varphi(k + 1))$, then we can just "*forget*" about $\varphi(1)$ and recover regular induction.

Okay, so the form of the argument is valid; what they're doing is equivalent to ordinary weak induction. Nothing is wrong in the basis step of the induction either: it really is *true* that every collection containing only one person is monochromatic, and their argument in support of that *true* conclusion is valid.

¹⁰ This is not necessarily the *only* way to characterize this flaw. It could be reasonably argued that this argument is flawed due to a failure to consider an *additional (missing) basis step* regarding collections of size 2. If that *additional basis step* verified correctly and included, and if the inductive part were adjusted to start at 2 instead of 1, then this would be a valid argument.

¹¹ Being "*monochromatic*" is a universally quantified property about a collection: a collection A of people is monochromatic when "*for all a_1 in A , for all a_2 in A , the eye color of a_1 is the same as the eye color of a_2* " is *true*. Since universally quantified statements are always *true* in empty universes (e.g., because the negation, which is existential, must be false), every empty collection of people must be monochromatic.

In the inductive step, they fix an arbitrary natural number k and assume that every collection of k -many people is monochromatic, intending to then prove that every collection of $(k + 1)$ -many people is monochromatic. This inductive hypothesis is correctly stated, and this kind of reasoning—doing induction on the *size* of the collections—should make sense if you think about it. As long as the collections in this argument are actually arbitrary, then there should be no problem here.

The argument then proceeds by fixing an arbitrary collection of $(k + 1)$ -many people X —consisting of $p_0, p_1, \dots, p_{k-1}, p_k$ —and splitting it into a “left” fragment L and a “right” fragment R that overlap in the middle.

$$\begin{array}{ccccccc} & \underbrace{\hspace{10em}} & L & & \underbrace{\hspace{10em}} & & \\ p_0 & p_1 & p_2 & \cdots & p_{k-2} & p_{k-1} & p_k \\ p_0 & p_1 & p_2 & \cdots & p_{k-2} & p_{k-1} & p_k \\ & \underbrace{\hspace{10em}} & R & & \underbrace{\hspace{10em}} & & \end{array}$$

They then argue that, because L and R are both monochromatic by the *inductive hypothesis*, and because they share at least one person in common, and because L and R combined cover all of X , we must conclude that X is monochromatic. However, this argument has to work for *every* natural number k , and when $k = 1$ we have a problem!

$$\begin{array}{cc} & \underbrace{\hspace{1em}} \\ & L \\ p_0 & p_1 \\ & \underbrace{\hspace{1em}} \\ & R \end{array}$$

In this case, there is no one at the intersection of L and R , so the inference that X is monochromatic fails! This means that $\varphi(1) \not\Rightarrow \varphi(2)$, which is logically equivalent to $\varphi(1) \wedge \neg\varphi(2)$. We can demonstrate this explicitly with an example.

$$\begin{array}{c} \{ \text{red circle}, \text{blue circle} \} \\ \{ \text{red circle} \} \{ \text{blue circle} \} \end{array}$$

Since they fail to correctly show $\varphi(1) \Rightarrow \varphi(2)$, they have not actually demonstrated $\forall k(\varphi(k) \Rightarrow \varphi(k + 1))$ in their inductive step, and the whole argument then falls apart. There are no other significant flaws in the argument.