

Problem Set 2

Discrete Structures

Due on the 21st day of September of the year of our Lord 2026 at 11:59 pm

Justify each of your answers with an argument or a proof as appropriate.

Answers presented without proof or justification are worse than incorrect.

Axiom 0: Zero is a Natural Number.

$$\exists x(x = 0).$$

Axiom 1: Zero is the First Natural Number.

$$\forall x(\text{succ}(x) \neq 0).$$

Axiom 2: The Successor of a Natural Number is a Natural Number.

$$\forall x \exists y(\text{succ}(x) = y).$$

Axiom 3: Natural Numbers with Equal Successors are Equal.

$$\forall x \forall y(\text{succ}(x) = \text{succ}(y) \Rightarrow x = y).$$

Axiom 4: Schema of Weak Induction.

For any *wff* φ containing at most one free variable, the following proposition holds.

$$(\varphi(0) \wedge \forall k(\varphi(k) \Rightarrow \varphi(\text{succ}(k)))) \Rightarrow \forall x(\varphi(x))$$

- For each of the following questions, justify your answer with a formal proof.
 - Prove that 2 is a natural number.
 - Prove $\forall x(\text{succ}(x) = x + 1)$.
- Use weak induction to prove $\forall x \forall y(\text{succ}(x) + y = \text{succ}(x + y))$.¹
- In this problem, we will establish the remaining half of the equivalence between induction and universal quantification over the natural numbers.

Let φ be an arbitrary *wff* containing at most one free variable.
Prove that $\forall x(\varphi(x)) \Rightarrow (\varphi(0) \wedge \forall k(\varphi(k) \Rightarrow \varphi(\text{succ}(k))))$.
- For this question, we will be thinking about the axiom schema of induction more abstractly and informally—outside of the context of the theory of arithmetic. Consider the following argument, which attempts to “*prove*” that attempts to prove that everyone has the same eye color as everyone else.

In order to demonstrate that everyone has the same eye color, we will argue that, within every nonempty finite collection of people, every person in the collection has the same eye color as everyone else. We therefore proceed by induction on the size of the nonempty finite collection of people. Notice that empty collections take care of themselves: if Z is an empty collection of people, then all of the people in Z clearly have the same eye color.

For clarity, we define the predicate $\mu(x, y) :=$ “ x has the same eye color as y ” for this argument.

Basis Step:

Let X be an arbitrary collection of people of size 1, so that X contains exactly one person. Let p be one of the people in X . We can clearly see that $\mu(p, x)$ is *true* for all x in X because p is

Some things to remember:

- succ is the *successor functional*, a symbol that, when given a natural number x as input, returns “the next number after x ” as its output, which we denote by writing $\text{succ}(x)$.
- 0 is our special constant symbol that refers to a distinguished object in the universe of discourse.
- We define the standard numerals by $1 := \text{succ}(0)$, $2 := \text{succ}(1)$, $3 := \text{succ}(2)$, and so on forever.
- Addition is defined recursively for all natural numbers n and m by:

$$\begin{aligned} n + 0 &:= n \\ n + \text{succ}(m) &:= \text{succ}(n + m) \end{aligned}$$

¹ There are two universally quantified variables in this expression. At the beginning of your proof, you should introduce x as an arbitrary natural number (in order to set up a use of *Universal Introduction* at the end). Once x is introduced (so it is a fixed constant), your task becomes proving $\forall y(\text{succ}(x) + y = \text{succ}(x + y))$, which you should do by induction.

the only person in X and $\mu(p, p)$ is obviously *true*. Thus, everyone in X has the same eye color. Therefore, all the people in any collection of size 1 have the same eye color.

Inductive Step:

Let k be a positive natural number. Assume that any collection of exactly k -many people all have the same eye color. We now need to prove that any collection of exactly $(k + 1)$ -many people all have the same eye color. Let X be a collection of people of size $k + 1$, so that the people inside X are $p_0, p_1, \dots, p_{k-1}, p_k$, so that each of these symbols refer to distinct people in X . We will now define two new subcollections of X .

Let L be the collection containing $p_0, \dots, p_{k-1}$. Let R be the collection containing $p_1, \dots, p_k$. Notice that L is a collection of exactly k -many people, so each of those people have the same eye color because of the assumption we made in our *inductive hypothesis*. Formally, this means $\forall i \forall j ((0 \leq i < k \wedge 0 \leq j < k) \Rightarrow \mu(p_i, p_j))$. Similarly, R is a collection of exactly k -many people, so each of the people in R have the same eye color by the same reasoning, meaning $\forall i \forall j ((1 \leq i < k + 1 \wedge 1 \leq j < k + 1) \Rightarrow \mu(p_i, p_j))$. It now follows that $\mu(p_0, p_1)$ and $\mu(p_1, p_k)$ are both *true*. Therefore, $\forall i \forall j ((0 \leq i < k + 1 \wedge 0 \leq j < k + 1) \Rightarrow \mu(p_i, p_j))$, meaning all of the people in X have the same eye color.

Since X was arbitrary, that means any collection of exactly $(k + 1)$ -many people all have the same eye color, as was required.

Now, we have shown that any collection with exactly one person contains people all with the same eye color (this also applies to collections with zero people). We have also shown that, for every natural number k , if every collection of k -many people all have the same eye color, then every collection of $(k + 1)$ -many people all have the same eye color. This is exactly what the schema of weak induction requires of us. Therefore, we can apply the schema and conclude that, for any natural number x , any collection of exactly x -many people all have the same eye color.

Therefore, everyone has the same eye color.

Since there are people in the world with different eye colors,² this argument is obviously not correct. Find and explain all of the flaws in this argument.

² E.g., the instructor has dark brown eyes, but his father has green eyes.