Problem Set 9

Discrete Structures

Due on the 12th day of May of the year of our Lord 2026 at 11:59 pm

As always, you may rely on any statement we have previously proven in lectures and problem sets. You should solve the problems *in order;* solutions to earlier problems may be applied as theorems in the proofs of later problems, but *not vice versa.*

Recall that a graph G is defined by a *finite* set of vertices[1] $V(G)$ and a set of edges $E(G)$ satisfying $E(G) \subseteq \{W \mid W \subseteq V(G) \wedge |W| = 2\}$. For any vertex $v \in V(G)$, we called $N_G(v) := \{w \in V(G) \mid \{v, w\} \in E(G)\}$ the *neighborhood* and $I_G(v) := \{e \in E(G) \mid v \in e\}$ *incidence set* of v in G respectively. The degree function $\mathtt{deg}_G : V(G) \to \mathbb{N}$ given by $\mathtt{deg}_G(v) := |I_G(v)|$ defines the *degree* of a node v by the number of edges incident on it.

[1] … also called *nodes*…

Definition: Walks and Paths.
Let G be a graph. A *"walk"* along G is meant to denote a possible way of visiting the vertices of G by traveling along the edges. Given a natural number $k \in \mathbb{N}$, we define a *walk of length k on G* to be a function $f : (k+1) \to V(G)$ such that, for each $i < k$, we have $\{f(i), f(i+1)\} \in E(G)$.[2] A *path* is a walk that does not repeat any nodes; more formally, a *path of length k on G* is an injective walk of length k on G.[3]

[2] This means that consecutive nodes $f(i)$ and $f(i+1)$ along the walk must be connected by an edge in G.

[3] Notice the *length* of a walk or path is determined by the number of *edges* involved, not the number of *vertices.*

Definition: Circuits and Cycles.
Given $k \in \mathbb{N}$, a *circuit of length k* on a graph G is a walk $w : k+1 \to V(G)$ that starts and ends at the same place, meaning $w(0) = w(k)$. If a circuit does not repeat any nodes other than the first and last ones,[4] then we call this circuit a *cycle*.

[4] … meaning the walk is essentially a *path* except for the fact that the first and last nodes are the same…

Definition: Connectivity and Trees.
A graph is said to be *connected* precisely when every distinct pair of nodes is connected by a path. Formally, given a graph G, we say *G is connected iff* the following is satisfied.

$$(\forall v, w \in V(G))(\exists k \in \mathbb{N})(\exists p : k+1 \to V(G))\big(p \text{ is a path} \wedge p(0) = v \wedge p(k) = w\big)$$

Any graph that is connected and does not contain any cycles is called a *tree*. Given a tree T and a node $v \in V(T)$, we say v is a *leaf iff* $\mathtt{deg}_T(v) = 1$; any vertex that is not a leaf is usually called an *internal* node.[5]

[5] In other contexts, you will encounter the notion of a *rooted* tree, which is a tree containing a distinguished node called *"the root"* of the tree. In those cases, the *height* of a tree is defined to be the length of the longest path between the root node and any leaf in the tree.

1. Show that, for every tree T, there exists a *unique* path between any two vertices in T.
2. Prove that every tree T satisfying $|V(T)| \geqslant 2$ must contain at least one leaf node.
3. Use induction to prove that every nonempty tree T satisfies $|V(T)| = |E(T)| + 1$.
4. Prove any graph G satisfying $(\forall v \in V(G))\left(\mathtt{deg}_G(v) \geqslant \frac{|V(G)|-1}{2}\right)$ is connected.