

Problem Set 8

Discrete Structures

Due on the 29th day of April of the year of our Lord 2026 at 11:59 pm

As always, you may rely on any statement we have previously proven in lectures and problem sets. You should solve the problems *in order*; solutions to earlier problems may be applied as theorems in the proofs of later problems, but *not vice versa*.

When you *define a function* in this problem set, you are *not* required to *prove* what you've defined is in fact a function (as long as you *have* actually defined a function).

Definition: Binomial Coefficients.

Let $n \in \mathbb{N}$ and consider an arbitrary finite set \mathcal{A} of cardinality $|\mathcal{A}| = n$. We will represent by $\binom{n}{k}$ the number of subsets of \mathcal{A} of cardinality k . We define this formally below.

$$\binom{n}{k} := |\{\mathcal{B} \mid \mathcal{B} \subseteq \mathcal{A} \wedge |\mathcal{B}| = k\}| = |\{\mathcal{B} \mid \mathcal{B} \subseteq n \wedge |\mathcal{B}| = k\}|$$

You may rely on the axiom of *trichotomy of cardinality*¹ and the following theorems.

¹For any A, B , either $|A| < |B|$, or $|A| > |B|$, or $|A| = |B|$.

Theorem.

$$\forall A \forall B (|A| \leq |B| \Leftrightarrow |B| \geq |A|).$$

Theorem.

For any sets \mathcal{F} and \mathcal{J} , if \mathcal{F} is finite and \mathcal{J} is infinite, then $|\mathcal{F}| < |\mathcal{J}|$.

Theorem.

\mathbb{N} is infinite. Additionally, $|\mathbb{N}| = |\mathbb{N}_+| = |\mathbb{Z}| = |\mathbb{Q}|$ and $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{Z} \times \mathbb{Z}|$.

Corollary.

For every $n \in \mathbb{N}$, we have $|\mathbb{N}| = |\mathbb{N} \setminus \{k \in \mathbb{N} \mid k < n\}|$.

1. An archaeologist on a recent expedition to the British isles has discovered, for each positive natural number $n \in \mathbb{N}_+$, evidence of a manuscript written in a language called *nglish*—believed to be used by the native inhabitants of *ngland* in ancient times. Each such manuscript is exactly n pages long and contains precisely $2n$ distinct words. Prove each manuscript contains a page with two distinct words on it.
2. Show that there exist $a, b \in \mathbb{N}$ such that $256 < a < b$ and $7^a \equiv 7^b \pmod{512}$.
3. Consider the set $\mathcal{S} := \{3, 4, 7, 8, 9, 10, 12, 15, 18, 19, 27, 28\}$. Show that, in any subset $\mathcal{X} \subseteq \mathcal{S}$ of cardinality $|\mathcal{X}| \geq 9$, there exist three *distinct* elements $x_1, x_2, x_3 \in \mathcal{X}$ such that $x_1 + x_2 + x_3 = 40$.
4. For this problem, you may assume that $\binom{n}{0} = \binom{n}{n} = 1$ for all $n \in \mathbb{N}$. Prove that, for every $n, k \in \mathbb{N}$ such that $k \leq n$, we have $\binom{n}{k} = \binom{n}{n-k}$.
5. We define a *finite binary string* to be a function $b : n \rightarrow \{0, 1\}$, where $n \in \mathbb{N}$ is the length of the string. We define an *infinite binary string* to be a function $b : \mathbb{N} \rightarrow \{0, 1\}$. How many infinite binary strings are there that each contain one contiguous finite substring consisting only of 1s? Formalize your claim and justify it with a proof.