

## Problem Set 5

### Discrete Structures

Due on the 23<sup>rd</sup> day of March of the year of our Lord 2026 at 11:59 pm

As always, you may rely on any statement we have previously proven in lectures and problem sets. You should solve the problems *in order*; solutions to earlier problems may be applied as theorems in the proofs of later problems, but *not vice versa*.

Recall that we have proven the following theorems about natural number arithmetic.

**Lemma.**

$$(\forall x \in \mathbb{N})(\text{suc}(x) \neq 0) \text{ and } (\forall x \in \mathbb{N})(x \neq 0 \Rightarrow (\exists y \in \mathbb{N})(\text{suc}(y) = x)).$$

**Lemma.**

$$(\forall x \in \mathbb{N})(\text{suc}(x) = x + 1) \text{ and } (\forall x \in \mathbb{N})(\text{suc}(x) = 1 + x).$$

**Theorem: Zero is the Additive Identity.**

$$(\forall x \in \mathbb{N})(x = x + 0) \text{ and } (\forall x \in \mathbb{N})(x = 0 + x).$$

**Theorem: Associativity of Addition.**

$$(\forall x, y, z \in \mathbb{N})(x + (y + z) = (x + y) + z).$$

Use the theorem of *weak induction* to prove each of the following theorems.

1. We say that a set  $x$  is  *$\in$ -transitive*  $:\Leftrightarrow \forall y(y \in x \Rightarrow \forall z(z \in y \Rightarrow z \in x))$ .  
Prove that every natural number is  $\in$ -transitive.
2. We will work up to a proof of commutativity of addition.
  - a. Prove  $(\forall x, y \in \mathbb{N})(\text{suc}(x + y) = \text{suc}(x) + y)$ .
  - b. Prove  $(\forall x, y \in \mathbb{N})(x + y = y + x)$ .
3. Show that  $(\forall x, y \in \mathbb{N})(x < y \Rightarrow (\exists n \in \mathbb{N})(x + n = y))$ .
4. Prove that  $(\forall x, y, z \in \mathbb{N})(x \cdot (y + z) = (x \cdot y) + (x \cdot z))$ .
5. Given a sequence of natural numbers  $f : \mathbb{N} \rightarrow \mathbb{N}$  and a natural number  $a \in \mathbb{N}$ , we recursively define the *iterated summation* of finitely many values of  $f$  as follows.

$$\sum_{i=a}^a f(i) := f(a)$$

$$\sum_{i=a}^{\text{suc}(b)} f(i) := \left( \sum_{i=a}^b f(i) \right) + f(\text{suc}(b)) \quad \text{where } b \in \mathbb{N} \text{ such that } a \leq b$$

For completeness, we say  $\sum_{i=a}^b f(i) := 0$  when  $b \in \mathbb{N}$  such that  $b < a$ . For this problem, you may also assume *associativity and commutativity of multiplication* over  $\mathbb{N}$ .

$$\text{Prove that } (\forall n \in \mathbb{N}) \left( 1 + \sum_{i=0}^n 2^i = 2^{n+1} \right).$$

Recall the definitions of *addition* and *multiplication*. Given  $n \in \mathbb{N}$  and  $m \in \mathbb{N}$ , we make the following definitions:

$$\begin{cases} n + 0 & := n \\ n + \text{suc}(m) & := \text{suc}(n + m) \end{cases}$$

$$\begin{cases} n \cdot 0 & := 0 \\ n \cdot \text{suc}(m) & := n \cdot m + n \end{cases}$$

Given  $n \in \mathbb{N}$  and  $m \in \mathbb{N}$ , we also recursively define *exponentiation* as follows:

$$\begin{cases} n^0 & := 1 \\ n^{\text{suc}(m)} & := n^m \cdot n \end{cases}$$