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Practice Exam 2

Discrete Structures

16th day of November of the year of our Lord 2026

1 *Answer the questions below by selecting exactly one answer for each.*

1. If X is finite, then $\forall Y((Y \subseteq X \ \wedge \ Y \neq X) \Rightarrow |Y| < |X|)$.
   - ● *True*
   - ○ *False*

   sets X with this property are called "Dedekind finite." Every Dedekind finite set is finite, and vice-versa.

   } A set X with the negation of this property is called "Dedekind infinite," and this is equivalent to being infinite.

2. If X is a set such that $(\forall x \in X)(\exists n \in \mathbb{N})(|n| = |x|)$, then X is finite.
   - ○ *True*
   - ● *False*

   $X := \mathbb{N}$ has this property (every individual natural number is a finite set), but we have already proven $\mathbb{N}$ is infinite.

3. If $f : X \to Y$ is *injective* and $g : X \to Y$ is *surjective*, then $|X| = |Y|$.
   - ● *True*
   - ○ *False*

   Since g is surjective, g has an injective right inverse $\tilde{g}: Y \to X$. Since $f: X \to Y$ and $g: Y \to X$ are both injective, the Cantor-Schröder-Bernstein theorem tells us there is a bijection from X to Y, so $|X| = |Y|$.

4. If $f : X \to Y$ is *injective* and $g : X \to Y$ is *surjective*, then f is a *bijection*.
   - ○ *True*
   - ● *False*

   there is nothing to suggest that f must be surjective.

5. For every $k \in \mathbb{N}$, there is a connected graph G where $(\forall v \in V(G))(\deg_G(v) = k)$.
   - ● *True*
   - ○ *False*

   For each $k \in \mathbb{N}$, the complete graph K_{k+1} has this property.

6. If G is a graph with k components and $(\forall v \in V(G))(\deg_G(v) \geqslant k)$, then $k = 1$.
   - ○ *True*
   - ● *False*

   this is a graph with 2 connected components and every vertex has degree equal to 2.

7. $\forall x(\cap x \subseteq \cup x)$.
   - ○ *True*
   - ● *False*

   $\cap\emptyset$ does not exist; even if it did, $\cap\emptyset \notin \cup\emptyset$ because $\emptyset \in \cap\emptyset$ but $\emptyset \notin \cup\emptyset = \emptyset$.

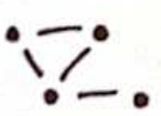
   $\forall x(x \in \cap\emptyset = \{z \mid \forall w(w \in \emptyset \Rightarrow z \in w)\})$.

8. $\cup(\mathbb{P}(x) \setminus \{\emptyset, x\}) = x$ for every set x.
   - ○ *True*
   - ● *False*

   let $X := \{\emptyset\}$. Then, $\mathbb{P}(X) = \mathbb{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\} = \{\emptyset, X\}$, so $\mathbb{P}(X) \setminus \{\emptyset, X\} = \mathbb{P}(\{\emptyset\}) \setminus \{\emptyset, \{\emptyset\}\} = \emptyset$.

   therefore, $\cup(\mathbb{P}(X) \setminus \{\emptyset, X\}) = \cup\emptyset = \emptyset \neq \{\emptyset\} = X$.

9. Every non-acyclic graph has at least three distinct nodes with the same degree.
   - ○ *True*
   - ● *False*

   this graph contains a cycle, but its nodes have degrees 3, 2, 2, and 1.

10. There are infinitely many subsets of $\mathbb{N}$ that are neither finite nor cofinite in $\mathbb{N}$.
    - ● *True*
    - ○ *False*

    let $E := \{x \in \mathbb{N} \mid x \text{ is even}\}$. $E, E \setminus 3, E \setminus 5, E \setminus 7, \ldots$ is an infinite sequence of subsets of $\mathbb{N}$ that are each individually infinite and also [not] cofinite in $\mathbb{N}$; from this, we know the set of all infinite non-cofinite subsets of $\mathbb{N}$ must be infinite (it contains an infinite subset).

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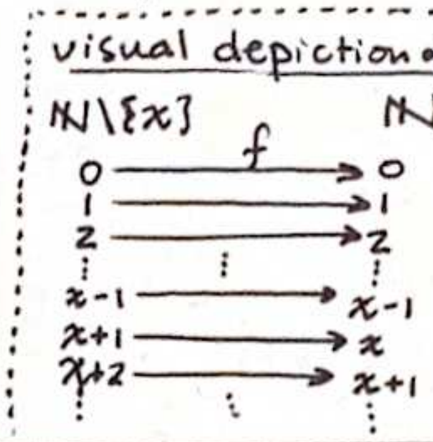
2 *Answer each of the questions below without writing a proof.*

1. Let $x \in \mathbb{N}$ and define $X := \mathbb{N} \setminus \{x\}$. Give an example of a bijection $f : X \to \mathbb{N}$.

Consider the function $f: \mathbb{N}\setminus\{x\} \to \mathbb{N}$ given by, for each $n \in \mathbb{N}\setminus\{x\}$,

$$f(n) := \begin{cases} n & \text{if } n < x \\ n-1 & \text{if } n > x \end{cases}$$

visual depiction of f



2. If $n \in \mathbb{N}$, how many cycles of length 3 does the complete graph K_n have?

Every ordered triple (u,v,w) where $u,v,w \in V(K_n)$ and $u \neq v$, $v \neq w$, and $u \neq w$ defines a cycle $f_{u,v,w}: C_3 \hookrightarrow K_n$ by sending $f_{u,v,w}(0) := u$, $f_{u,v,w}(1) := v$, and $f_{u,v,w}(2) := w$. Different ordered triples define different monomorphisms (meaning different cycles), and each cycle is induced by such an ordered triple (they are in bijection with each other), so to count the cycles we can instead count the ordered triples made from distinct vertices. .....

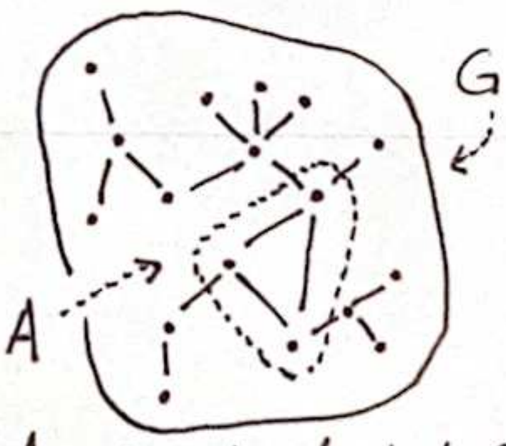
→ Given an ordered triple (u,v,w) as described, there are $\boxed{n}$ different choices of vertex for u. Then, since we can't pick u again, there are $\boxed{n-1}$ choices for v for each choice of u. Similarly, since we can't pick u or v again, there are $\boxed{n-2}$ choices for w for each u and v. So, there are $\boxed{n(n-1)(n-2)}$ total

3. Suppose G is a graph with $n := \#V(G)$ and $m := \#E(G)$.
How many induced subgraphs does G contain?

Each subset $W \subseteq V(G)$ induces a subgraph $G[W]$; each induced subgraph is (by definition) induced by a subset of $V(G)$, and different subsets induce different subgraphs because they have different vertex sets. So, the set of induced subgraphs of G is in bijection with $\mathbb{P}(V(G))$.
Since $\#\mathbb{P}(V(G)) = 2^n$, there are $\boxed{2^n}$ induced subgraphs in G.

4. Let G be a connected graph. Suppose there is a unique $A \subseteq G$ with the following property: there exists $k \in \mathbb{N} \setminus \{0,1,2\}$ such that A is isomorphic to C_k.
How many distinct spanning trees does G contain?

G is connected and contains exactly one subgraph isomorphic to a cycle — C_k, for some $k \in \mathbb{N}\setminus\{0,1,2\}$. Removing an edge that lies on a cycle in a connected graph maintains connectivity; therefore, removing any of the k edges that lie in the subgraph A would destroy the only cycle in G but maintain the connectivity, so it would produce a subtree of G. Since such a subtree would still use all of the vertices of G, this would be a spanning tree for G. Thus, the removal of any of these k edges produces a spanning tree for G, each spanning tree can be expressed as G after removing only one such edge (because otherwise, you would disconnect the subgraph), and removing different edges produces different spanning trees, so the set of spanning trees for G is in bijection with $E(A)$.



An example of what G might look like.

→ Since $\#E(A) = k$, there are exactly $\boxed{k}$ spanning trees for G.

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3 *For this problem, you may rely on any result that we have discussed so far.*

Let $f : \mathbb{N} \to \mathbb{N}$ be an arbitrary function, and consider the function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined recursively below for each pair of natural numbers $n \in \mathbb{N}$ and $x \in \mathbb{N}$.

$$\varphi(0, x) := x$$
$$\varphi(n+1, x) := f\big(\varphi(n, x)\big)$$

1. a. Suppose φ is injective. Must f be injective? Justify your answer with a proof.

Sure! or not! whatever! it doesn't matter!... because φ is not injective!
To see why, notice that, for any $x \in \mathbb{N}$, $\begin{cases} \varphi(0, f(x)) = f(x) \\ \varphi(1, x) = f(\varphi(0, x)) = f(x) \end{cases}$,
so that $\varphi(0, f(x)) = f(x) = \varphi(1, x)$, but $(0, f(x)) \neq (1, x)$ because their first coordinates are not equal.
In particular, $\varphi(0, f(0)) = f(0) = \varphi(1, 0)$, but $(0, f(0)) \neq (1, 0)$, so φ is not injective.

b. Suppose f is injective. Must φ be injective? Justify your answer with a proof.

No! Consider $f := \mathrm{id}_{\mathbb{N}}$, so that $f(n) = n$ for all $n \in \mathbb{N}$. This is an injection (in fact, a bijection) as we have discussed. However, notice what happens to φ:

for any $x \in \mathbb{N}$, for any $n \in \mathbb{N}$, $\varphi(n+1, x) = f(\varphi(n, x)) = \varphi(n, x)$, but $(n+1, x) \neq (n, x)$.

In particular, $\varphi(1, 0) = f(\varphi(0, 0)) = \varphi(0, 0)$, but $(1, 0) \neq (0, 0)$. So, φ is not injective.

2. a. Suppose f is surjective. Must φ be surjective? Justify your answer with a proof.

Yes! In fact, φ was already surjective regardless of whether f was surjective or not.
To see this, let $y \in \mathbb{N}$ and notice that $(0, y) \in \mathbb{N} \times \mathbb{N}$ and $\varphi(0, y) = y$.
Therefore, φ is surjective.

b. Suppose φ is surjective. Must f be surjective? Justify your answer with a proof.

Clearly, since φ is surjective regardless of f, we can simply pick a non-surjective f in order to conclude No!

Consider $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) := 0$ for all $x \in \mathbb{N}$. f is clearly not surjective since $(\forall x \in \mathbb{N})(f(x) = 0 \neq 1)$ and $1 \in \mathbb{N}$.

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4 *For this problem, you may rely on any result that we have discussed so far.*

Show that every graph (typo: ← with at least one node!) contains a path of maximal length.

A graph with no nodes does not contain any paths of any length, so it would not contain a longest path.

Let G be a graph such that $V(G) \neq \emptyset$. Recall that $V(G)$ is finite, so we know there exists $n \in \mathbb{N}$ such that $|n| = |V(G)|$. Since every bijection has a (unique) two-sided inverse, and this inverse is also a bijection, we know $|V(G)| = |n|$, so there exists a bijection $f: V(G) \to n$. Now, let $K \in \mathbb{N}$ and consider an arbitrary path $p: P_K \hookrightarrow G$ of length K in G. Notice that $p: V(P_K) \to V(G)$, the underlying vertex map, is an injection by definition. Composing these two injections produces an injective function $f \circ p: V(P_K) \to n$. Recalling that $V(P_K) = K+1$, and that $K+1 = \{z \mid z \in \mathbb{N} \wedge z < K+1\}$ and $n = \{z \mid z \in \mathbb{N} \wedge z < n\}$, this tells us $|K+1| \leq |n|$ by definition, so that $K+1 \leq n$. We know $n > 0$ because $V(G) \neq \emptyset$, so $K \leq n-1$. Therefore, every path in G has length no greater than $n-1$.

Further, since $V(G) \neq \emptyset$, there is some vertex $v \in V(G)$. The function $p_v: \{0\} \to V(G)$ given by $p_v(0) := v$ is obviously an injective graph homomorphism from P_0 into G, and so there is at least one path in G (in particular, a path of length 0).

Consider the set $\mathcal{L} := \{K \in \mathbb{N} \mid (\exists p: P_K \hookrightarrow G)(p \text{ is a path in } G)\}$ of all lengths of paths in G. Notice $0 \in \mathcal{L}$, so $\mathcal{L} \neq \emptyset$, and that $(\forall l \in \mathcal{L})(l \leq n-1)$, meaning $\mathcal{L}$ is bounded from above. We will now prove $\mathcal{L}$ has a greatest element:

As a result of this proof, you now have this theorem: every nonempty, bounded above subset of $\mathbb{N}$ has a greatest element!

Define the set $M := \{m \in \mathbb{N} \mid (\forall l \in \mathcal{L})(l < m)\}$, and notice $n \in M$, so $M \neq \emptyset$. Also, $M \subseteq \mathbb{N}$. Therefore, M contains a least element m^*, meaning $(\forall m \in M)(m^* \leq m)$ and $m^* \in M$. Since $0 \in \mathcal{L}$ and $0 < m^*$, we can deduce that $m^*-1 \in \mathbb{N}$, $m^*-1 \notin M$, and $m^*-1 \in \mathcal{L}$. Since $(\forall l \in \mathcal{L})(l < m^*)$, we now know $(\forall l \in \mathcal{L})(l \leq m^*-1)$; this, combined with $m^*-1 \in \mathcal{L}$, is what it means for m^*-1 to be the greatest element of $\mathcal{L}$.

So, we now know $\mathcal{L}$ has a greatest element; call it "m^*-1". Then, there must be some path $p^*: P_{m^*-1} \hookrightarrow G$ of length m^*-1 in G, and there are no longer paths since m^*-1 is the greatest element of $\mathcal{L}$. Therefore, G contains a path of maximal length.

QED

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5 *For this problem, you may rely on any result that we have discussed so far.*

Suppose you know that, for any tree T and for any node $v \in V(T)$, the induced subgraph $T[V(T) \setminus \{v\}]$ has exactly $\deg_T(v)$ distinct connected components.

Recall that every graph contains a node of maximal degree.
For any graph G, define $\Delta(G)$ to be the maximum degree in G.
Prove that every tree T contains at least $\Delta(T)$ leaves.[1][2]

[1] *Hint:* it may help to try a proof by induction, though this is not strictly necessary.

[2] *Hint:* what happens if you remove a node with maximal degree?

Let T be a tree, and let $v \in V(T)$ such that $\Delta T = \deg_T(v)$, which means $(\forall w \in V(T))(\deg_T(w) \leq \deg_T(v))$.

Consider the induced subgraph $R := T[V(T) \setminus \{v\}]$ formed by removing the node v along with its incident edges from T. Let $C := \{H \mid H \text{ is a connected component of } R\}$. Observe the following.

Let $w \in N_T(v)$ and $\tilde{w} \in N_T(v)$ such that $w \neq \tilde{w}$. We can see that $\{w, v\} \in E(T)$ and $\{\tilde{w}, v\} \in E(T)$ because w and $\tilde{w}$ are neighbors of v in T, so $\{(0,w), (1,v), (2,\tilde{w})\}$ is a path in T from w to $\tilde{w}$. Since T is a tree, this is the only path in T from w to $\tilde{w}$. Since $R \subseteq T$, $v \notin V(R)$, and v is a vertex along the path in T from w to $\tilde{w}$, we can see that there is no path in R from w to $\tilde{w}$ (if there were, then there would be a path in T from w to $\tilde{w}$ that didn't involve the vertex v, contradicting the uniqueness of the path $\{(0,w), (1,v), (2,\tilde{w})\}$ in T). Since connected components are connected, this means $(\forall H \in C)(w \in V(H) \Rightarrow \tilde{w} \notin V(H))$; w and $\tilde{w}$ can not lie on the same connected component of R.

Further, let $u \in N_T(v)$, [redacted] and let $\mathcal{U} \subseteq R$ be the subgraph given by $V(\mathcal{U}) := \{u\}$ and $E(\mathcal{U}) := \emptyset$. Note that $\mathcal{U}$ is obviously connected. Now, consider the set of all connected subgraphs of R containing u, which we define by $\mathcal{Y} := \{H \subseteq R \mid H \text{ is connected} \wedge u \in V(H)\}$. Notice $(\forall y \in \mathcal{Y})(\#V(y) \leq \#V(R))$, [redacted] and $(\forall y \in \mathcal{Y})(\#E(y) \leq \#E(R))$, so the sets of natural numbers corresponding to node-counts and edge counts of elements of $\mathcal{Y}$ are both bounded from above; [redacted] thus, each of those sets has a greatest element. For the same reason, the set of natural numbers corresponding to [redacted] edge-counts of elements of $\mathcal{Y}$ that have the greatest node-counts also has a greatest element. This corresponds to some [redacted] subgraph of R that is connected, contains u, and has as many nodes and edges as possible while maintaining those two properties, so this is a connected component by definition. Therefore, $(\exists H \in C)(u \in V(H))$.

Showing $(\forall r \in V(R))(\exists H \in C)(r \in V(H))$ implies $(\forall u \in N_T(v))(\exists H \in C)(u \in V(H))$. Showing $(\forall H_1, H_2 \in C)(V(H_1) \cap V(H_2) = \emptyset)$ implies that two neighbors of v can't share the same connected component. Both of these subproofs follow directly from the fact that the set of connected components in a graph partitions the set of vertices (and edges) of that graph.

If we know this partitioning theorem, then these subproofs are not necessary.

Therefore, the function $f : N_T(v) \to C$ that maps $w \in N_T(v)$ to the component $f(w) \in C$ that contains the vertex w is an injection, so $|N_T(v)| \leq |C|$.

Now, notice that each $H \in C$ is connected (by definition) and acyclic (otherwise, the cycle in H would be a cycle in T, contradicting the fact that T is a tree), so each $H \in C$ is a tree.
For each $H \in C$, we now note that $V(H) \neq \emptyset$ due to maximality of connected components and observe:

I. If $\#V(H) = 1$, then $V(H) = \{h\}$ for some $h \in V(R)$, and thus $\deg_R(h) = 0$. If $\{h, v\} \notin E(T)$, then T would be disconnected, so $\{h, v\} \in E(T)$, and thus $\deg_T(h) = 1$.

II. If $\#V(H) \geq 2$, then H contains two leaves $t_1, t_2 \in V(H)$ with $\deg_R(t_1) = 1 = \deg_R(t_2)$. Since t_1 is connected to t_2 through a path, either $\{t_1, v\} \notin E(T)$ or $\{t_2, v\} \notin E(T)$; otherwise, if they were both adjacent to v, T would contain a cycle (also, they would be on the same connected component, contradicting our earlier result). Therefore, either $\deg_T(t_1) = 1$ or $\deg_T(t_2) = 1$, since [redacted] $E(T) \setminus E(R) = \delta_T(v)$.

From this, we now have [redacted] $(\forall H \in C)(\exists h \in V(H))(\deg_T(h) = 1)$.

Therefore,
$$\begin{aligned}
\#\{h \in V(T) \mid \deg_T(h) = 1\} &\geq \#\{h \in V(T) \mid (\exists H \in C)(h \in V(H) \wedge \deg_T(h) = 1)\} \\
&\geq \#C \\
&\geq \#N_T(v) \\
&= \deg_T(v) \\
&= \Delta(T),
\end{aligned}$$

So we conclude that T contains at least $\Delta(T)$ leaves.

QED