NAME: NETID:

Practice Exam 2

Discrete Structures

16th day of November of the year of our Lord 2026

1 *Answer the questions below by selecting exactly one answer for each.*

1. If X is finite, then $\forall Y((Y \subseteq X \ \wedge \ Y \neq X) \Rightarrow |Y| < |X|)$.

   ○ *True*
   ○ *False*

2. If X is a set such that $(\forall x \in X)(\exists n \in \mathbb{N})(|n| = |x|)$, then X is finite.

   ○ *True*
   ○ *False*

3. If $f : X \to Y$ is *injective* and $g : X \to Y$ is *surjective*, then $|X| = |Y|$.

   ○ *True*
   ○ *False*

4. If $f : X \to Y$ is *injective* and $g : X \to Y$ is *surjective*, then f is a *bijection*.

   ○ *True*
   ○ *False*

5. For every $k \in \mathbb{N}$, there is a connected graph G where $(\forall v \in V(G))(\texttt{deg}_G(v) = k)$.

   ○ *True*
   ○ *False*

6. If G is a graph with k components and $(\forall v \in V(G))(\texttt{deg}_G(v) \geqslant k)$, then $k = 1$.

   ○ *True*
   ○ *False*

7. $\forall x(\cap x \subseteq \cup x)$.

   ○ *True*
   ○ *False*

8. $\cup(\mathbb{P}(x) \setminus \{\varnothing, x\}) = x$ for every set x.

   ○ *True*
   ○ *False*

9. Every non-acyclic graph has at least three distinct nodes with the same degree.

   ○ *True*
   ○ *False*

10. There are infinitely many subsets of $\mathbb{N}$ that are neither finite nor cofinite in $\mathbb{N}$.

    ○ *True*
    ○ *False*

2 *Answer each of the questions below without writing a proof.*

1. Let $x \in \mathbb{N}$ and define $X := \mathbb{N} \setminus \{x\}$. Give an example of a bijection $f : X \to \mathbb{N}$.

2. If $n \in \mathbb{N}$, how many cycles of length 3 does the complete graph K_n have?

3. Suppose G is a graph with $n := \#V(G)$ and $m := \#E(G)$.
How many induced subgraphs does G contain?

4. Let G be a connected graph. Suppose there is a unique $A \subseteq G$ with the following property: there exists $k \in \mathbb{N} \setminus \{0, 1, 2\}$ such that A is isomorphic to C_k.
How many distinct spanning trees does G contain?

3 *For this problem, you may rely on any result that we have discussed so far.*

Let $f : \mathbb{N} \to \mathbb{N}$ be an arbitrary function, and consider the function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined recursively below for each pair of natural numbers $n \in \mathbb{N}$ and $x \in \mathbb{N}$.

$$\varphi(0, x) := x$$
$$\varphi(n + 1, x) := f\big(\varphi(n, x)\big)$$

1. a. Suppose φ is injective. Must f be injective? Justify your answer with a proof.

   b. Suppose f is injective. Must φ be injective? Justify your answer with a proof.

2. a. Suppose f is surjective. Must φ be surjective? Justify your answer with a proof.

   b. Suppose φ is surjective. Must f be surjective? Justify your answer with a proof.

4 *For this problem, you may rely on any result that we have discussed so far.*

Show that every graph with at least one vertex contains a path of maximal length.

5 *For this problem, you may rely on any result that we have discussed so far.*

Suppose you know that, for any tree T and for any node $v \in V(T)$, the induced subgraph $T[V(T) \setminus \{v\}]$ has exactly $\texttt{deg}_T(v)$ distinct connected components.

Recall that every graph contains a node of maximal degree.
For any graph G, define $\Delta(G)$ to be the maximum degree in G.
Prove that every tree T contains at least $\Delta(T)$ leaves.[1][2]

[1] *Hint:* it may help to try a proof by induction, though this is not strictly necessary.

[2] *Hint:* what happens if you remove a node with maximal degree?