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Practice Exam 1

Discrete Structures

28th day of September of the year of our Lord 2026

1 *Answer the questions below by selecting exactly one answer for each.*

1. Let φ be a *wff*. Which of the following are equivalent to $\forall n(\varphi(n))$?[1]

   - ● $\varphi(0) \wedge \forall k(\varphi(k) \Rightarrow \varphi(k+1))$
   - ● $\varphi(0) \wedge \neg\exists k(\varphi(k) \wedge \neg\varphi(k+1))$
   - ○ $\forall k(\forall j(j < k \Rightarrow \varphi(j)) \Rightarrow \varphi(k+1))$
   - ● $\forall k(\forall j(j < k \Rightarrow \varphi(j)) \Rightarrow \varphi(k))$

   $\equiv \forall k(\neg(\varphi(k) \wedge \neg\varphi(k+1)))$
   $\equiv \forall k(\neg\varphi(k) \vee \varphi(k+1))$
   $\equiv \forall k(\varphi(k) \rightarrow \varphi(k+1))$.

[1] This question has *three* valid answers.

2. Which of the following propositions ***contradicts*** the axioms of arithmetic?

   - ○ $\exists x\forall y(\mathrm{suc}(y) \neq x)$ 0 has no predecessors;
   - ○ $\forall x\exists y\exists z(x + z = y)$
   - ● $\forall x\exists y(\mathrm{suc}(y) = x)$ $x=0$ contradicts this
   - ○ $\exists x\forall y\exists z(x + z = y)$

3. If p and q are propositions, which of the following can be inferred from $p \vee q$?

   - ○ $p \rightarrow \neg q$
   - ○ p
   - ○ $\neg p \wedge \neg q$
   - ● $\neg q \rightarrow p$

   $p \vee q \equiv q \vee p \equiv \neg\neg q \vee p \equiv \neg q \rightarrow p$

4. If p and q are propositions, which of the following is equivalent to $\neg(p \rightarrow q)$?

   - ● $p \wedge \neg q$
   - ○ $\neg p \rightarrow \neg q$
   - ○ $p \vee \neg q$
   - ○ $\neg q \rightarrow \neg p$

   $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \equiv \neg\neg p \wedge \neg q \equiv p \wedge \neg q$

5. If φ is a *wff* such that $\forall k(\varphi(k) \Rightarrow \varphi(k+2))$, which numbers must φ be *true* about?

   - ○ *The even natural numbers*
   - ○ *All of the natural numbers*
   - ○ *None of the natural numbers*
   - ● *Not enough information*

6. This sentence is *false.*

   - ○ *True*
   - ● *Neither True nor False*
   - ○ *False*
   - ○ *Not enough information*

7. If this sentence is *false,* then $\exists m(3 = 2^m)$.

   - ○ *True*
   - ○ *Neither True nor False*
   - ○ *False*
   - ● *Not enough information*

8. If 0 is a multiple of 2, then this sentence is *false.*

   - ○ *True*
   - ● *Neither True nor False*
   - ○ *False*
   - ○ *Not enough information*

9. In the theory of arithmetic, the symbol $\mathrm{suc}(\bigcirc)$ denotes a predicate.

   - ○ *True*
   - ○ *Neither True nor False*
   - ● *False*
   - ○ *Not enough information*

10. Strong induction is capable of proving more things than weak induction.

    - ○ *True*
    - ○ *Neither True nor False*
    - ● *False*
    - ○ *Not enough information*

$$\left(\varphi(0) \wedge \forall k(\varphi(k) \rightarrow \varphi(k+1))\right) \Leftrightarrow \forall k(\forall j(j<k \Rightarrow \varphi(j)) \rightarrow \varphi(k)).$$

they are equivalent

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2 *Follow the instructions for each question below.*

1. Find a closed form for the recurrence relation given below. Your closed form should not involve iterated operations. You do not need to justify your answer.

$$T(1) := 4$$
$$T(2n) := 2T(n) + 2n$$

$\forall n\,(\exists m(n=2^m) \Rightarrow T(n) = n\cdot\log_2(n) + 4\cdot n)$.

$T(n) = 2\cdot T(n/2) + n$
$= 2(2\cdot T((n/2)/2) + n/2) + n$
$= 2^2\cdot T(n/2^2) + 2n$
$= 2^2(2\cdot T((n/2^2)/2) + n/2^2) + 2n$
$= 2^3\cdot T(n/2^3) + 3n$
$\vdots$
$= 2^k\cdot T(n/2^k) + k\cdot n$
$\vdots$
$n/2^k = 1 \Leftrightarrow k = \log_2(n)$
$\vdots$
$= 2^{\log_2(n)}\cdot T(n/2^{\log_2(n)}) + \log_2(n)\cdot n$
$= n\cdot T(1) + n\cdot\log_2(n)$
$= 4\cdot n + n\cdot\log_2(n)$.

2. Let φ be a *wff* and consider the two propositions given below.

$$\forall k\big(\forall j(j < k \Rightarrow \varphi(j)) \;\Rightarrow\; \varphi(k)\big)$$
$$\forall k\big(\forall j(j \leqslant k \Rightarrow \varphi(j)) \;\Rightarrow\; \varphi(k+1)\big)$$

Explain why the first line is equivalent to $\forall n(\varphi(n))$ but the second line is not.

It should be clear that $\forall n(\varphi(n))$ implies both lines above; use universal elimination to prove $\varphi(k)$ (first line) or $\varphi(k+1)$ (second line). Let's look at the converse direction. Assuming $\forall k(\forall j(j<k \Rightarrow \varphi(j)) \Rightarrow \varphi(k))$, then we know in particular, "when $k=0$", that $\forall j(\underline{j<0 \Rightarrow \varphi(j)}) \Rightarrow \varphi(0)$. Since $\forall x(x \geq 0)$, the premise of this inner conditional is false, which means $\forall j(j<0 \Rightarrow \varphi(j))$ is true; applying Modus Ponens then gives us $\boxed{\varphi(0)}$, even though it seems like that sentence is doing induction "without a basis step." Then, "when $k=1$", we have $\forall j(j<1 \Rightarrow \varphi(j)) \Rightarrow \varphi(1)$, and in this case the only value of j satisfying the premise of the inner conditional is 0, so we can simplify the statement to $(0<1 \Rightarrow \varphi(0)) \Rightarrow \varphi(1)$, which simplifies to $\varphi(0) \Rightarrow \varphi(1)$. Since we already verified $\varphi(0)$, Modus Ponens again tells us $\boxed{\varphi(1)}$. Continuing on in this way, we can prove $\varphi(n)$ for every natural number n.

The second line says $\forall k(\forall j(j \leq k \Rightarrow \varphi(j)) \Rightarrow \varphi(k+1))$, and "when $k=0$", this yields $\forall j(j \leq 0 \Rightarrow \varphi(j)) \Rightarrow \varphi(0+1)$, which simplifies to $\varphi(0) \Rightarrow \varphi(1)$. "when $k=1$", it simplifies to $(\varphi(0) \wedge \varphi(1)) \Rightarrow \varphi(2)$. All of these are conditional statements, but we don't know the premise of any of them because we can't derive $\varphi(0)$! So we can't prove $\varphi(n)$ for any n!

When $k=2$, we have: $\forall j(j<2 \Rightarrow \varphi(j)) \Rightarrow \varphi(2)$, which simplifies to $(\varphi(0) \wedge \varphi(1)) \Rightarrow \varphi(2)$. Since we know both $\varphi(0)$ and $\varphi(1)$, we then get $\varphi(2)$.

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3 *For this problem, you may rely on any result of arithmetic we have studied. You may carry out calculations involving addition, multiplication, and exponentiation of specific numbers without justification.*

We define a natural number x to be *"even"* when $\exists y(x = 2 \cdot y)$. We similarly define a natural number x to be *"odd"* when $\exists y(x = 2 \cdot y + 1)$.

Prove that $n \cdot (n + 1)$ is even for every natural number n.

Gauss' Little Theorem

Proof 1: Let n be an arbitrary natural number, and recall the theorem $2 \cdot \sum_{i=0}^{n} i = n \cdot (n+1)$. Since n is a natural number, we know $\sum_{i=0}^{n} i$ is a natural number, and therefore $\exists m(2 \cdot m = n \cdot (n+1))$. this shows $n \cdot (n+1)$ is even by definition. QED

Proof 2: We will prove $\forall n(\exists m(2m = n \cdot (n+1)))$ by induction.

Basis Step: We will show $\exists m(2m = 0 \cdot (0+1))$.

Observe $0 \cdot (0+1) = 0 \cdot (1) = 0 = 2 \cdot 0$. therefore, $\exists m(2 \cdot m = 0 \cdot (0+1))$.

Inductive Step: Let k be an [redacted] arbitrary natural number.

Assume $\exists m(2m = k \cdot (k+1))$. this means there exists some j such that $2j = k \cdot (k+1)$.

We need to show $\exists m(2m = (k+1) \cdot ((k+1)+1))$. So, like "always", let's observe.

$$
\begin{aligned}
(k+1) \cdot ((k+1)+1) &= (k+1)(k+(1+1)) && \text{by associativity of addition} \\
&= (k+1)(k+2) \\
&= (k+1) \cdot k + (k+1) \cdot 2 && \text{by distributivity of multiplication over addition} \\
&= k \cdot (k+1) + 2 \cdot (k+1) && \text{by commutativity of multiplication} \\
&= (2 \cdot j) + 2 \cdot (k+1) && \text{by the inductive hypothesis} \\
&= 2 \cdot (j + (k+1)) && \text{by distributivity of multiplication over addition}
\end{aligned}
$$

Since j and k are natural numbers, $(j + (k+1))$ is a natural number, and therefore $\exists m(2 \cdot m = (k+1) \cdot ((k+1)+1))$ as desired.

Therefore, we conclude $\forall n(\exists m(2m = n \cdot (n+1)))$ by the axiom schema of weak induction, which means $n \cdot (n+1)$ is even for every natural number by definition. QED

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4 *For this problem, you may rely on any result of arithmetic we have studied. You may carry out calculations involving addition, multiplication, and exponentiation of specific numbers without justification.*

Given two arbitrary natural numbers x and y, we say that "*x divides y*" when we can express y as a multiple of x—we define this formally below.

$$x \mid y \;:\Leftrightarrow\; \exists m(x \cdot m = y)$$

Use any valid form of induction to show that $4 \mid 9^n + 3$ for every natural number n.

Proof: We will prove $\forall n(4 \mid 9^n+3)$ by strong induction (because why not).

Let K be an arbitrary natural number and assume the following:

$\forall j(j<K \Rightarrow (4 \mid 9^j+3))$ ← this is our inductive hypothesis.

We will now prove $4 \mid 9^K+3$. We take two cases.

Case 1: Suppose $K=0$. Then, $9^0+3 = 1+3 = 4 = 4\cdot 1$, so $\exists m(4\cdot m = 9^0+3)$. This shows $4 \mid 9^0+3$ by definition.

Case 2: Suppose $K \neq 0$, which implies there exists j such that $K = j+1$ thanks to a theorem. We now observe:

$$\begin{aligned} 9^K+3 = 9^{j+1}+3 &= 9^j\cdot 9+3 && \text{by definition of exponentiation} \\ &= 9^j\cdot(8+1)+3 \\ &= (9^j\cdot 8 + 9^j\cdot 1)+3 && \text{by distributivity of multiplication over addition} \\ &= 9^j\cdot 8 + (9^j\cdot 1+3) && \text{by associativity of addition} \\ &= 9^j\cdot 8 + (9^j+3) && \text{because } \forall x(x\cdot 1 = x). \end{aligned}$$

Now, notice that $K=j+1$ implies $K>j$ by definition, so by the inductive hypothesis we can say $4\cdot \ell = 9^j+3$ for some natural number ℓ. Then, we see:

$$\begin{aligned} (9^j\cdot 8)+(9^j+3) &= 9^j\cdot 8 + 4\ell && \text{by the inductive hypothesis} \\ &= (4\cdot 2)\cdot 9^j + 4\ell && \text{by commutativity of multiplication} \\ &= 4\cdot(2\cdot 9^j) + 4\cdot\ell && \text{by associativity of multiplication} \\ &= 4\cdot(2\cdot 9^j+\ell) && \text{by distributivity of multiplication over addition} \end{aligned}$$

Since j and ℓ are natural numbers, $2\cdot 9^j+\ell$ is a natural number, so $\exists m(4m = 9^K+3)$, which means $4 \mid 9^K+3$ by definition.

Therefore, we conclude that $4 \mid 9^K+3$ as desired, showing $\forall n(4 \mid 9^n+3)$ by the theorem of strong induction.

QED

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5 *For this problem, you may only assume the statements written on the side.*

Prove that multiplication distributes over addition by demonstrating the following.

$$\forall x \forall y \forall z (x \cdot (y + z) = (x \cdot y) + (x \cdot z))$$

In this problem, you may assume all of the axioms of the theory of arithmetic.

0. $\exists x(x = 0)$.
1. $\forall x(\mathrm{suc}(x) \neq 0)$
2. $\forall x \exists y(y = \mathrm{suc}(x))$
3. $\forall x \forall y(\mathrm{suc}(x) = \mathrm{suc}(y) \Rightarrow x = y)$
4. The Schema of Weak Induction

Additionally, you may assume the following theorems about addition.

5. $\forall x(x + 1 = \mathrm{suc}(x))$
7. $\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$
8. $\forall x \forall y(x + y = y + x)$

You may also assume the following theorems about multiplication.

9. $\forall x(0 \cdot x = 0)$
10. $\forall x(x \cdot 1 = x) \wedge \forall x(1 \cdot x = x)$

Proof: Let x and y be natural numbers. We will prove $\forall z(x \cdot (y+z) = (x \cdot y) + (x \cdot z))$ by (weak) induction.

Basis Step: We will show $x \cdot (y+0) = (x \cdot y) + (x \cdot 0)$.

Observe
$$\begin{aligned} x \cdot (y+0) &= x \cdot y && \text{by definition of addition} \\ &= (x \cdot y) + 0 && \text{by definition of addition} \\ &= (x \cdot y) + (x \cdot 0) && \text{by definition of multiplication} \end{aligned}$$

Thus, $x \cdot (y+0) = (x \cdot y) + (x \cdot 0)$.

Inductive Step: Let k be an arbitrary natural number.

Assume $x(y+k) = (x \cdot y) + (x \cdot k)$. We will now show $x \cdot (y + \mathrm{suc}(k)) = (x \cdot y) + (x \cdot \mathrm{suc}(k))$.

Observe.

$$\begin{aligned} x \cdot (y + \mathrm{suc}(k)) &= x \cdot \mathrm{suc}(y + k) && \text{by definition of addition.} \\ &= x \cdot (y+k) + x && \text{by definition of multiplication} \\ &= (x \cdot y + x \cdot k) + x && \text{by the inductive hypothesis} \\ &= x \cdot y + (x \cdot k + x) && \text{by associativity of addition} \\ &= x \cdot y + x \cdot \mathrm{suc}(k) && \text{by definition of multiplication} \end{aligned}$$

Therefore, $x \cdot (y + \mathrm{suc}(k)) = (x \cdot y) + (x \cdot \mathrm{suc}(k))$ as required.

We therefore conclude $\forall a \forall b \forall c(a \cdot (b+c) = (a \cdot b) + (a \cdot c))$ by the axiom schema of weak induction.

QED