

Practice Exam 1

Discrete Structures

28th day of September of the year of our Lord 2026

1 Answer the questions below by selecting exactly one answer for each.

1. Let φ be a wff. Which of the following are equivalent to $\forall n(\varphi(n))$?¹

¹ This question has *three* valid answers.

- ☐ $\varphi(0) \wedge \forall k(\varphi(k) \Rightarrow \varphi(k+1))$
☐ $\varphi(0) \wedge \neg \exists k(\varphi(k) \wedge \neg \varphi(k+1))$
☐ $\forall k(\forall j(j < k \Rightarrow \varphi(j)) \Rightarrow \varphi(k+1))$
☐ $\forall k(\forall j(j < k \Rightarrow \varphi(j)) \Rightarrow \varphi(k))$

2. Which of the following propositions **contradicts** the axioms of arithmetic?

- ☐ $\exists x \forall y(\text{succ}(y) \neq x)$
☐ $\forall x \exists y \exists z(x + z = y)$
☐ $\forall x \exists y(\text{succ}(y) = x)$
☐ $\exists x \forall y \exists z(x + z = y)$

3. If p and q are propositions, which of the following can be inferred from $p \vee q$?

- ☐ $p \rightarrow \neg q$
☐ p
☐ $\neg p \wedge \neg q$
☐ $\neg q \rightarrow p$

4. If p and q are propositions, which of the following is equivalent to $\neg(p \rightarrow q)$?

- ☐ $p \wedge \neg q$
☐ $\neg p \rightarrow \neg q$
☐ $p \vee \neg q$
☐ $\neg q \rightarrow \neg p$

5. If φ is a wff such that $\forall k(\varphi(k) \Rightarrow \varphi(k+2))$, which numbers must φ be true about?

- ☐ The even natural numbers
 ☐ All of the natural numbers
☐ None of the natural numbers
 ☐ Not enough information

6. This sentence is false.

- ☐ True
 ☐ Neither True nor False
☐ False
 ☐ Not enough information

7. If this sentence is false, then $\exists m(3 = 2^m)$.

- ☐ True
 ☐ Neither True nor False
☐ False
 ☐ Not enough information

8. If 0 is a multiple of 2, then this sentence is false.

- ☐ True
 ☐ Neither True nor False
☐ False
 ☐ Not enough information

9. In the theory of arithmetic, the symbol $\text{succ}(\odot)$ denotes a predicate.

- ☐ True
 ☐ Neither True nor False
☐ False
 ☐ Not enough information

10. Strong induction is capable of proving more things than weak induction.

- ☐ True
 ☐ Neither True nor False
☐ False
 ☐ Not enough information

2 Follow the instructions for each question below.

1. Find a closed form for the recurrence relation given below. Your closed form should not involve iterated operations. You do not need to justify your answer.

$$\begin{aligned}T(1) &:= 4 \\T(2n) &:= 2T(n) + 2n\end{aligned}$$

2. Let φ be a *wff* and consider the two propositions given below.

$$\begin{aligned}\forall k(\forall j(j < k \Rightarrow \varphi(j)) &\Rightarrow \varphi(k)) \\ \forall k(\forall j(j \leq k \Rightarrow \varphi(j)) &\Rightarrow \varphi(k+1))\end{aligned}$$

Explain why the first line is equivalent to $\forall n(\varphi(n))$ but the second line is not.

- 3 For this problem, you may rely on any result of arithmetic we have studied. You may carry out calculations involving addition, multiplication, and exponentiation of specific numbers without justification.

We define a natural number x to be “even” when $\exists y(x = 2 \cdot y)$. We similarly define a natural number x to be “odd” when $\exists y(x = 2 \cdot y + 1)$.

Prove that $n \cdot (n + 1)$ is even for every natural number n .

- 4 For this problem, you may rely on any result of arithmetic we have studied. You may carry out calculations involving addition, multiplication, and exponentiation of specific numbers without justification.

Given two arbitrary natural numbers x and y , we say that “ x divides y ” when we can express y as a multiple of x —we define this formally below.

$$x \mid y :\Leftrightarrow \exists m(x \cdot m = y)$$

Use any valid form of induction to show that $4 \mid 9^n + 3$ for every natural number n .

5 For this problem, you may only assume the statements written on the side.

Prove that multiplication distributes over addition by demonstrating the following.

$$\forall x \forall y \forall z (x \cdot (y + z) = (x \cdot y) + (x \cdot z))$$

In this problem, you may assume all of the axioms of the theory of arithmetic.

- 0. $\exists x (x = 0)$.
- 1. $\forall x (\text{succ}(x) \neq 0)$
- 2. $\forall x \exists y (y = \text{succ}(x))$
- 3. $\forall x \forall y (\text{succ}(x) = \text{succ}(y) \Rightarrow x = y)$
- 4. The Schema of Weak Induction

Additionally, you may assume the following theorems about addition.

- 5. $\forall x (x + 1 = \text{succ}(x))$
- 7. $\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$
- 8. $\forall x \forall y (x + y = y + x)$

You may also assume the following theorems about multiplication.

- 9. $\forall x (0 \cdot x = 0)$
- 10. $\forall x (x \cdot 1 = x) \wedge \forall x (1 \cdot x = x)$