Solutions for Practice Midterm 2

Discrete Structures

1st day of May of the year of our Lord 2026

1 *Answer each of the following questions by marking True or False but not both.*

1. If X and Y are finite and $f : X \to Y$ is a surjection, then f is injective.

   ○ *True*
   ● *False*

2. $\forall x \forall y \forall z \big((x \setminus \{z\} = y \setminus \{z\}) \Rightarrow (x = y) \big)$.

   ○ *True*
   ● *False*

3. For any set X, the mapping $\{(x, \{x\}) \mid x \in X\}$ is a function from X to $\mathbb{P}(X)$.

   ● *True*
   ○ *False*

4. If X and Y are sets and $\lambda : X \to Y$, then $|\lambda| = |X|$.

   ● *True*
   ○ *False*

5. $\forall x \exists y \big(|x| > |y| \big)$.

   ○ *True*
   ● *False*

6. If A and B are sets such that $A \subseteq B$ and $A \neq \varnothing$, then $|B \setminus A| < |B|$.

   ○ *True*
   ● *False*

7. It is possible to construct a graph G such that $(\forall v \in V(G))(\mathtt{deg}_G(v) = 1)$.

   ● *True*
   ○ *False*

8. It is possible to construct a graph G with n nodes and 2^n edges for every $n \in \mathbb{N}$.

   ○ *True*
   ● *False*

9. The set of all finite binary strings is uncountably infinite.

   ○ *True*
   ● *False*

10. If X and Y satisfy $|X| = |Y|$ and $f : X \to Y$ is an injection, then f is also a surjection.

    ○ *True*
    ● *False*

2 *You may assume all axioms and theorems we have discussed so far, including any basic arithmetic and algebraic properties of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.*

Consider the recursively defined function $f : \{2^x \mid x \in \mathbb{N}\} \to \mathbb{N}$ described below.

$$f(n) := \begin{cases} 1 & \text{if } n = 1 \\ 2 \cdot f\left(\frac{n}{2}\right) + n & \text{if } n > 1 \end{cases}$$

Prove that $f(n) = n \cdot \log_2(n) + n$ for all $n \in \{2^x \mid x \in \mathbb{N}\}$.

Proof. We will prove $(\forall n \in \mathbb{N})\big(f(2^n) = 2^n \log_2(2^n) + 2^n\big)$ by weak induction.

Basis Step:
Observe that $f\left(2^0\right) = f(1) = 1 = 1 \cdot 0 + 1 = 2^0 \cdot \log_2\left(2^0\right) + 2^0$.

Inductive Step:
Let $k \in \mathbb{N}$ and assume $f\left(2^k\right) = 2^k \cdot \log_2\left(2^k\right) + 2^k$. We now observe the following.

$$\begin{aligned}
f\left(2^{k+1}\right) &= 2 \cdot f\left(\frac{2^{k+1}}{2}\right) + 2^{k+1} && \text{by the definition of } f \\
&= 2 \cdot f\left(\frac{2 \cdot 2^k}{2}\right) + 2^{k+1} \\
&= 2 \cdot f\left(2^k\right) + 2^{k+1} \\
&= 2 \cdot \left(2^k \cdot \log_2\left(2^k\right) + 2^k\right) + 2^{k+1} && \text{by the } \textit{inductive hypothesis} \\
&= 2 \cdot 2^k \cdot \log_2\left(2^k\right) + 2 \cdot 2^k + 2^{k+1} \\
&= 2^{k+1} \cdot \log_2\left(2^k\right) + 2^{k+1} + 2^{k+1} \\
&= 2^{k+1}\left(\log_2\left(2^k\right) + 1\right) + 2^{k+1} \\
&= 2^{k+1}\left(\log_2\left(2^k\right) + \log_2\left(2^1\right)\right) + 2^{k+1} \\
&= 2^{k+1} \cdot \log_2\left(2^k \cdot 2\right) + 2^{k+1} \\
&= 2^{k+1} \cdot \log_2\left(2^{k+1}\right) + 2^{k+1}
\end{aligned}$$

Therefore, $f\left(2^{k+1}\right) = 2^{k+1} \cdot \log_2\left(2^{k+1}\right) + 2^{k+1}$ as desired.

We therefore conclude that $f(2^n) = 2^n \cdot \log_2(2^n) + 2^n$ for all $n \in \mathbb{N}$.

Now, for each $n \in \{2^x \mid x \in \mathbb{N}\}$, we know there exists $x \in \mathbb{N}$ such that $n = 2^x$. We just proved that $f(2^x) = 2^x \cdot \log_2(2^x) + 2^x$. That now means $f(n) = n \cdot \log_2(n) + n$ as was initially required.

QED

3 *You may assume all axioms and theorems we have discussed so far, including any basic arithmetic and algebraic properties of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.*

Consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $f(x,y) := 3^x 5^y$ for $(x,y) \in \mathbb{N} \times \mathbb{N}$. Prove that f is injective.

Proof. Let $(a,b),(x,y) \in \mathbb{N} \times \mathbb{N}$ and suppose $f(a,b) = f(x,y)$, so $3^a 5^b = 3^x 5^y$. Towards a contradiction, suppose $(a,b) \neq (x,y)$, so that $a \neq x$ or $b \neq y$. We take two cases.

Case 1:
Assume $a \neq x$; without loss of generality, suppose $a < x$. This implies $x - a > 0$, so that $x - a - 1 \geqslant 0$ in particular. If we recall that multiplication on $\mathbb{N}$ is cancellative, we can observe the following.

$$3^a 5^b = 3^x 5^y \;\Rightarrow\; 5^b = 3^{x-a} 5^y \;\Rightarrow\; 5^b = 3\left(3^{x-a-1} 5^y\right)$$

Since $x - a - 1 \in \mathbb{N}$ and $y \in \mathbb{N}$, we know $3^{x-a-1} 5^y \in \mathbb{Z}$, so that $3 \mid 5^b$. Because 3 is prime, Euclid's lemma implies $3 \mid 5$, contradicting the fact that 5 is prime. ↯

Case 2:
Assume $b \neq y$; without loss of generality, suppose $b < y$. This implies $y - b > 0$, so that $y - b - 1 \geqslant 0$ in particular. Again, recalling that multiplication is cancellative over $\mathbb{N}$, we can make the following observation.

$$3^a 5^b = 3^x 5^y \;\Rightarrow\; 3^a = 3^x 5^{y-b} \;\Rightarrow\; 5^b = 5\left(3^x 5^{y-b-1}\right)$$

Since $y - b - 1 \in \mathbb{N}$ and $x \in \mathbb{N}$, we know $3^x 5^{y-b-1} \in \mathbb{Z}$, so that $5 \mid 3^a$. Because 5 is prime, Euclid's lemma tells us $5 \mid 3$, implying $5 \leqslant 3$ and contradicting $3 < 5$. ↯

QED

4 *You may assume all axioms and theorems we have discussed so far, including any basic arithmetic and algebraic properties of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.*

Prove that there are an even number of odd-degree nodes in any graph.

Proof. Let G be a graph, and let us define the sets $\varepsilon := \{v \in V(G) \mid 2 \mid \mathtt{deg}_G(v)\}$ and $\omega := \{v \in V(G) \mid 2 \nmid \mathtt{deg}_G(v)\}$ to respectively represent the sets of even-degree and odd-degree vertices in G. Notice that $V(G) = \varepsilon \cup \omega$, and also $\varepsilon \cap \omega = \varnothing$.

Since $V(G)$ is finite, we know $E(G)$ is finite, so that ε and ω are both finite sets. This implies there exist $n, m \in \mathbb{N}$ such that $|\varepsilon| = |n|$ and $|\omega| = |m|$, from which we know there exist two bijections $\varphi : n \to \varepsilon$ and $\psi : m \to \omega$ by definition. We will use these bijections as a helpful way of enumerating the elements of ε and ω; specifically, we now know that $\varepsilon = \{\varphi(i) \mid 0 \leqslant i < n\}$ and $\omega = \{\psi(j) \mid 0 \leqslant j < m\}$.

Further, since every element of ε has even degree, we will define $k_i \in \mathbb{Z}$ to be the integer such that $\mathtt{deg}_G(\varphi(i)) = 2k_i$ for each $i \in n$. Similarly, we will define $\ell_j \in \mathbb{Z}$ to be the integer such that $\mathtt{deg}_G(\psi(j)) = 2\ell_j + 1$ for each of the vertices in ω, which we know have odd degree. We can now recall that $\sum_{v \in V(G)} \mathtt{deg}_G(v) = 2 \cdot |E(G)|$ and observe.

$$\begin{aligned} 2 \cdot |E(G)| = \sum_{v \in V(G)} \mathtt{deg}_G(v) &= \sum_{v \in \varepsilon} \mathtt{deg}_G(v) + \sum_{v \in \omega} \mathtt{deg}_G(v) \\ &= \sum_{i=0}^{n-1} \mathtt{deg}_G(\varphi(i)) + \sum_{j=0}^{m-1} \mathtt{deg}_G(\psi(j)) \\ &= \sum_{i=0}^{n-1} 2k_i + \sum_{j=0}^{m-1} 2\ell_j + 1 \\ &= \sum_{i=0}^{n-1} 2k_i + \sum_{j=0}^{m-1} 2\ell_j + \sum_{j=0}^{m-1} 1 \end{aligned}$$

This then implies the following.

$$2\left(|E(G)| - \sum_{i=0}^{n-1} k_i - \sum_{j=0}^{m-1} \ell_j\right) = \sum_{j=0}^{m-1} 1 = m = |\omega|$$

Therefore, $2 \mid |\omega|$, showing us that there are an even number of odd-degree nodes in G.

QED